Inflation’s Role in Optimal Monetary-Fiscal Policy∗

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Abstract

We study how the maturity structure of nominal government debt affects optimal monetary and fiscal policy decisions and outcomes in a conventional new Keynesian model with a distorting tax. Key findings are: there is always a role for current and future inflation innovations to revalue government debt, reducing reliance on distorting taxes; the role of inflation in optimal fiscal financing increases with the average maturity of government debt; inflation is relatively more important as a fiscal shock absorber in high-debt than in low-debt economies; in some calibrations that are relevant to U.S. data, welfare is higher under the fully optimal monetary and fiscal policies than under the conventional optimal monetary policy with passively adjusting lump-sum taxes.

Keywords: tax smoothing, debt management, debt maturity

JEL Codes: E31, E52, E62, E63

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1 Introduction

Many countries have designed monetary and fiscal policy institutions that construct firm walls between the two policy authorities. There are good practical reasons for this separation: historically, high- or hyperinflation episodes have sprung from governments pressuring central banks to finance spending by printing high-powered money. Economic theory does not uniformly support the complete separation. If inflation is costless, as in neoclassical models with flexible wages and prices, then Chari and Kehoe (1999) show that an optimal policy generates jumps in inflation that revalue nominal government debt without requiring changes in distorting tax rates, much as inflation behaves under the fiscal theory of the price level [Leeper (1991), Sims (1994), Woodford (1995)].

Schmitt-Grohé and Uribe (2004) and Siu (2004) overturn this role for inflation with the striking result that even a modicum of price stickiness makes the optimal volatility of inflation close to zero, an outcome later confirmed by Kirsanova and Wren-Lewis (2012), among others. Out of this optimal policy literature has emerged the “current consensus assignment” for monetary and fiscal policy, which Kirsanova, Leith, and Wren-Lewis (2009) articulate: give monetary policy the task of controlling demand and inflation and fiscal policy the job of stabilizing debt. Actual policy arrangements in most countries are consistent with the literature’s conclusions.

Woodford (1998), Cochrane (2001) and Sims (2001, 2013) question the consensus assignment. They argue that with nominal government debt, adjustments in price levels revalue debt to absorb fiscal disturbances. This obviates the need to adjust distorting tax rates. When outstanding government debt has long maturity, it can be optimal to finance higher government spending with a little bit of inflation spread over the maturity of the debt, effectively converting nominal debt into state-contingent real debt, as in Lucas and Stokey (1983). Both Cochrane and Sims employ ad hoc welfare functions to illustrate their points, so neither argues that revaluation of debt through inflation is a feature of a fully optimal policy. Cochrane derives a theory of optimal inflation smoothing, while Sims asks how to optimally finance a one-time temporary increase in government spending with a mix of distorting taxes and inflation.

Benigno and Woodford (2007) examine optimal monetary policy when fiscal policy is suboptimal due to, for example, political constraints. They pose a different question than we do. They ask how a central bank should optimally target inflation in the face of political constraints that set distorting taxes or lump-sum transfers arbitrarily, rather than optimally.

1.1 What We Do We consider the canonical new Keynesian model that Benigno and Woodford (2004, 2007) examine in which the steady state is distorted by monopolistically
competitive firms and where nominal rigidities prevent firms from choosing new prices each period. A distorting tax is levied against firms’ sales. Total factor productivity and government purchases of goods fluctuate exogenously. Government issues nominal debt whose average duration is indexed by a single parameter. We focus on a model in which lump-sum transfers also fluctuate exogenously, but monetary and tax policies are chosen optimally to maximize welfare of the representative household. Optimal policies and the nature of resulting equilibria depend on both the maturity structure and level of government debt.\footnote{In a learning environment, Eusepi and Preston (2012) find that maturity structure and level of government debt have important consequences for stability.} We contrast welfare in this model to an alternative setup in which monetary policy is optimal, lump-sum taxes may be adjusted to ensure the government’s solvency condition never binds, but the distorting tax rate varies exogenously.

Table 1 reports the average term to maturity of outstanding government debt in selected advanced economies in recent years. While there is variation across countries, with Korea at 16 quarters and the United Kingdom at 49 quarters, the median is about six years. Figure 1 shows that the average duration of outstanding U.S. treasuries has varied substantially since 1951, reaching a high of six years in the early 1950s and a low of just two years in the mid-1970s.

<table>
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<tr>
<th>Country</th>
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Table 1: Average maturity of outstanding government debt, 1997–2010; Japan 1997–2009; South Korea 2001–2010. Source: OECD.
2 Model

We employ a standard new Keynesian economy that consists of a representative household with an infinite planning horizon, a collection of monopolistically competitive firms that produce differentiated goods, and a government. A fiscal authority finances exogenous expenditures with distorting taxes and debt and a monetary authority sets the short-term nominal interest rate.

2.1 Households The economy is populated by a continuum of identical households. Each household has preferences defined over consumption, $C_t$, and hours worked, $N_t$. Preferences are

$$E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$$

(1)

where $U(C_t, N_t)$ is given by

$$U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}$$
Consumption is defined over a basket of goods of measure one and indexed by \( j \)

\[
C_t = \left[ \int_0^1 C_{jt}^{\frac{1}{\epsilon}} dj \right]^{\frac{\epsilon}{1-\epsilon}}
\]

where \( C_{jt} \) represents the quantity of good \( j \) consumed by the household in period \( t \). The parameter \( \epsilon > 1 \) denotes the intratemporal elasticity of substitution across different varieties of consumption goods. When \( \epsilon \to \infty \), goods become perfect substitutes and the consumption index is linear; when \( \epsilon \to 1 \), the consumption index becomes Cobb-Douglas, goods are neither substitutes nor complements: an increase in the price of one good has no effect on demand for other goods.

Each good \( j \) is produced using a type of labor that is specific to that industry, \( N_{jt} \), the quantity of labor supply of type \( j \) in period \( t \). The representative household supplies all types of labor and aggregate labor supply is

\[
N_t = \left[ \int_0^1 N_{jt}^{1+\phi} dj \right]^{\frac{1}{1+\phi}}
\]

The aggregate price index \( P_t \) is

\[
P_t = \left[ \int_0^1 P_{jt}^{1-\epsilon} dj \right]^{\frac{1}{1-\epsilon}}
\]

where \( P_{jt} \) is the nominal price of the final goods produced in industry \( j \).

Households maximize (1) subject to the budget constraint

\[
C_t + Q_t^S \frac{B_t^S}{P_t} + Q_t^M \frac{B_t^M}{P_t} = B_{t-1}^S \frac{B_{t-1}^M}{P_t} + (1 + \rho Q_t^M) \frac{B_{t-1}^M}{P_t} + \int_0^1 \left( \frac{W_{jt}}{P_t} N_{jt} + \Pi_{jt} \right) dj + Z_t
\]

where \( W_{jt} \) is the nominal wage rate to \( j \)th industry, \( \Pi_{jt} \) denotes the share of profits paid by the \( j \)th industry to the households, and \( Z_t \) is lump-sum government transfer payments. \( B_t^S \) is a one-period government bond with nominal price \( Q_t^S \); \( B_t^M \) is a more general portfolio of government bond with price \( Q_t^M \). The general bond portfolio is defined as perpetuities with a constant coupon decay factor \( \rho \), as in Woodford (2001). Bonds issued at date \( t \) pay \( \rho^k \) dollars at date \( t + k + 1 \). The duration of the bond portfolio \( B_t^M \) is \( (1 - \beta \rho)^{-1} \). When \( \rho = 0 \), all bonds are one period and when \( \rho = 1 \), all bonds are consols.

\footnote{The bond’s yield to maturity is \( Q_t^M - (1 - \rho) \). At the risk of abusing the term, we shall refer to \( \rho \) as determining the average maturity of the portfolio.}
Household maximization yields the first-order conditions

\[
\frac{W_{jt}}{P_t} = -\frac{U_{n_{j,t}}}{U_{c,t}} \tag{2}
\]

\[
Q_t^S = \beta E_t \frac{U_{c,t+1}}{P_{t+1}} P_t \tag{3}
\]

\[
Q_t^M = \beta E_t \frac{U_{c,t+1}}{P_{t+1}} P_t (1 + \rho Q_{t+1}^M) \tag{4}
\]

Combining (3) and (4) yields the no-arbitrage condition between one-period and long-term bonds

\[
Q_t^M = E_t Q_t^S (1 + \rho Q_{t+1}^M) \tag{5}
\]

Iterating on (27) and imposing a terminal condition yields the term structure relation

\[
Q_t^M = \sum_{k=0}^{\infty} \rho^k E_t Q_{t+k}^S \tag{6}
\]

### 2.2 Firms

A continuum of monopolistically competitive firms produce differentiated goods. Production of good \( j \) is given by

\[
Y_{jt} = A_t N_{jt}
\]

where \( A_t \) is an aggregate technology shock, common across firms, which evolves exogenously.

Firm \( j \) faces the demand schedule

\[
Y_{jt} = \left( \frac{P_{jt}}{P_t} \right)^{-\epsilon} Y_t
\]

With demand imperfectly price-elastic, each firm has some market power, leading to the monopolistic competition distortion in the economy.

A second distortion stems from nominal rigidities. Price are staggered, as in Calvo (1983), with a fraction \( 1 - \theta \) of firms are permitted to choose a new price, \( P_t^* \), each period, while the remaining firms cannot adjust their prices. This pricing behavior implies the aggregate price index

\[
P_t = [(1 - \theta)(P_t^*)^{1-\epsilon} + \theta(P_{t-1})^{1-\epsilon}]^{\frac{1}{1-\epsilon}} \tag{7}
\]

Firms that can reset their price choose \( P_t^* \) to maximize the expected sum of discounted
future profits by solving

$$\max E_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k}[ (1 - \tau_{t+k}) P^*_{t+k} Y_{t+k|t} - \Psi_{t+k}(Y_{t+k|t}) ]$$

subject to the demand schedule

$$Y_{t+k|t} = \left( \frac{P^*_{t+k}}{P_{t+k}} \right)^{-\epsilon} Y_{t+k}$$

where $Q_{t,t+k}$ is the stochastic discount factor for the price at $t$ of one unit of composite consumption goods at $t + k$. Sales revenues are taxed at rate $\tau_t$, $\Psi_t$ is cost function, and $Y_{t+k|t}$ is output in period $t + k$ for a firm that last reset its price in period $t$.

The first-order condition for this maximization problem implies that the price in period $t$ satisfies

$$\frac{P^*_t}{P_t} = \frac{\epsilon}{\epsilon - 1} \frac{E_t \sum_{k=0}^{\infty} (\beta \theta)^k U_{c,t+k} Y_{t+k|t} \text{MC}_{t+k|t}(\frac{P_{t+k}}{P_t})^{\epsilon}}{E_t \sum_{k=0}^{\infty} (\beta \theta)^k (1 - \tau_{t+k}) U_{c,t+k} Y_{t+k|t} (\frac{P_{t+k}}{P_t})^{\epsilon-1}}$$

where $\text{MC}_{t+k|t}$ is real marginal cost at period $t + k$ for the firm that last reset its price in period $t$. Real marginal cost can be rewritten as

$$\text{MC}_{t+k|t} = \frac{1}{A_{t+k} U_{c,t+k}} \left( \frac{P^*_t}{P_{t+k}} \right)^{-\epsilon \phi} \left( \frac{Y_{t+k}}{A_{t+k}} \right)^{\phi}$$

Substituting (9) into (8), we obtain the expression

$$\left( \frac{P^*_t}{P_t} \right)^{1+\epsilon \phi} = \frac{\epsilon}{\epsilon - 1} \frac{E_t \sum_{k=0}^{\infty} (\beta \theta)^k (\frac{Y_{t+k}}{A_{t+k}})^{\phi+1} (\frac{P_{t+k}}{P_t})^{\epsilon(1+\phi)}}{E_t \sum_{k=0}^{\infty} (\beta \theta)^k (1 - \tau_{t+k}) U_{c,t+k} Y_{t+k|t} (\frac{P_{t+k}}{P_t})^{\epsilon-1}}$$

$$= \frac{\epsilon}{\epsilon - 1} K_t$$

where $K_t$ and $J_t$ are aggregate variables that satisfy the recursive relations

$$K_t = \left( \frac{Y_t}{A_t} \right)^{\phi+1} + \beta \theta E_t K_{t+1} \pi_{t+1}^{\epsilon(1+\phi)}$$

$$J_t = (1 - \tau_t) U_{c,t} Y_t + \beta \theta E_t J_{t+1} \pi_{t+1}^{\epsilon-1}$$
2.3 Government

The government consists of a monetary and a fiscal authority that face the consolidated budget constraint, expressed in real terms:

\[ Q_t^M \delta_t^M + S_t = (1 + \rho Q_t^M) \frac{b_{t-1}}{\pi_t} \]  

(13)

where the real primary budget surplus is

\[ S_t = \tau_t Y_t - Z_t - G_t \]

(14)

and \( G_t \) is exogenous government demand for the composite good.

An intertemporal equilibrium—or solvency—condition links the market value of outstanding government debt to the expected present value of primary surpluses:

\[
\left( E_t \sum_{k=0}^{\infty} \beta^k \frac{U_{c,t+k}}{U_{c,t}} \frac{\rho^k}{P_{t+k}} \right) B_{t-1}^M = E_t \sum_{k=0}^{\infty} \beta^k \frac{U_{c,t+k}}{U_{c,t}} S_{t+k}
\]

(15)

where \( \beta^k \frac{U_{c,t+k}}{U_{c,t}} \) is the \( k \)-period real discount factor.

Cochrane (2001) interprets expression (15) as a “bond valuation equation” that yields feasible price level sequences for any given expected present value of primary surpluses and initial face value of outstanding bonds, \( B_{t-1}^M \). The average maturity \( \rho \) and real discount rate give the rates at which the government can trade off the price level today for price levels in the future.

The central bank controls the riskless short-term nominal gross interest rate \( i_t \), which is related to other financial asset prices through the no-arbitrage condition

\[ i_t = [Q_t^S]^{-1} = \left[ \beta E_t \frac{U_{c,t+1}}{U_{c,t}} \frac{P_t}{P_{t+1}} \right]^{-1} \]

allowing (15) to be written as

\[
\left( 1 + E_t \sum_{k=1}^{\infty} \frac{\rho^k}{i_t \delta_{t+1} \ldots \delta_{t+k-1}} \right) \frac{B_{t-1}^M}{P_t} = E_t \sum_{k=0}^{\infty} \beta^k \frac{U_{c,t+k}}{U_{c,t}} S_{t+k}
\]

(16)

Equilibrium condition (16) reflects a fundamental symmetry between monetary and fiscal policies. The price level today must be consistent with expected future monetary and fiscal policies, whether those policies are set optimally or not. Bond maturity matters: so long

\( ^3 \)See Appendix A for the derivation of this condition.
as the average maturity exceeds one period, \( \rho > 0 \), expected future monetary policy in the form of choices of the short-term nominal interest rate, \( i_{t+k} \), plays a role in determining the current price level.

In the fully optimal policy problem, government chooses functions for the tax rate, \( \tau_t \), and the short-term nominal interest rate, \( i_t \), taking exogenous processes for technology, \( A_t \), government purchases, \( G_t \), and transfers, \( Z_t \), as given. We derive how the optimal policy and welfare vary with the average maturity of government debt, as indexed by \( \rho \). We examine the case of a distorted steady state. We focus on optimal policy commitment and adopt Woodford’s (2003) “timeless perspective.”

2.4 Equilibrium Market clearing in the goods market requires

\[
Y_t = C_t + G_t
\]

and market clearing in labor market requires

\[
\Delta_t^{-\gamma} Y_t = A_t N_t
\]

where \( \Delta_t = \int_0^1 (\frac{P_j}{P_t})^{-\epsilon(1+\phi)} dj \) denotes the measure of price dispersion across firms and satisfies the recursive relation

\[
\Delta_t = (1 - \theta) \left[ \frac{1 - \theta \pi_t^{(1+\varphi)}}{1 - \theta} \right]^{(1+\varphi)} \theta \Delta_t^{-\gamma} + \theta \pi_t^{(1+\varphi)} \Delta_{t-1}
\]

Price dispersion is the source of welfare losses from inflation variability.

3 Fully Optimal Policy

Appendices C–F detail the derivations underlying the linear-quadratic approximation to the government’s optimum problem. Because Benigno and Woodford (2004) originated these derivations, here we cut to the chase and begin with the linear-quadratic approximation.

3.1 Linear-Quadratic Approximation We compute a linear-quadratic approximation to the nonlinear optimal solutions, using the methods that Benigno and Woodford (2004) develop. This allows us to characterize the optimal policy responses to fluctuations in the exogenous disturbance processes within a neighborhood of the steady state.

Distorting taxes and monopolistic competition conspire to make the deterministic steady state inefficient, so the standard linear-quadratic approach does not yield an accurate ap-
proximation of the optimal policy.\textsuperscript{4} Benigno and Woodford (2004) show that a correct linear-quadratic approximation is still possible even in the case of a distorted steady state. Their approach computes a second-order approximation to the model’s structural equations and uses an appropriate linear combination of those equations to eliminate the linear terms in the second-order approximation to the welfare measure to obtain a purely quadratic expression. There are three advantages to the linear-quadratic approach: first, it allows us to obtain analytical results rather than purely numerical ones; second, it nests both conventional analyses of optimal monetary policy and analyses of optimal tax-smoothing; third, the welfare criterion is policy-independent, which can be used to rank alternative sub-optimal policies.

Welfare losses experienced by the representative household are, up to a second-order approximation, proportional to\textsuperscript{5}

\begin{equation}
\frac{1}{2}E_0 \sum_{t=0}^{\infty} \beta^t \left( q_\pi \hat{x}_t^2 + q_x \hat{x}_t^2 \right)
\end{equation}

where the weight on output stabilization depends on model parameters

\[
q_x \equiv \frac{\kappa}{\epsilon} \left[ 1 + \frac{s_c^{-1} \sigma}{\varphi + s_c^{-1} \sigma} \left( (1 + w_g)(1 + w_\tau) - s_c^{-1}(1 + w_g + w_\tau) \right) \right]
\]

\(\hat{x}_t\) denotes the welfare-relevant output gap, defined as the deviation between \(\hat{y}_t\) and its efficient level \(\hat{y}^e_t\): \(\hat{x}_t \equiv \hat{y}_t - \hat{y}^e_t\). Efficient output, \(\hat{y}^e_t\), depends on the three fundamental shocks and is given by \(\hat{y}^e_t = q_A \hat{A}_t + q_G \hat{G}_t + q_Z \hat{Z}_t\).\textsuperscript{6} \(w_g = (\bar{Z} + \bar{G})/\bar{S}\) is steady-state transfer payment plus government spending to surplus ratio, \(w_\tau = \bar{\tau}/1 - \bar{\tau}\), \(s_c = \bar{C}/\bar{Y}\) is the consumption to GDP ratio, and

\[
\kappa = \frac{(1 - \theta)(1 - \beta \theta)}{\theta} \frac{s_c^{-1} \sigma + \varphi}{1 + \epsilon \varphi}
\]

\[
\Gamma = (s_c^{-1} \sigma + \varphi)(1 + w_g) + s_c^{-1} \sigma w_\tau - w_\tau(1 + w_g)
\]

\[
\Phi = 1 - \epsilon \frac{1 - (1 - \bar{\tau})}{\epsilon}
\]

Noting that \(-\frac{U_c}{U_\pi} = (1 - \Phi)MPN\), \(\Phi\), which measures the inefficiency of the steady state, depends on the steady state tax rate, \(\bar{\tau}\), and the elasticity of substitution between differentiated goods, \(\epsilon\). Under the assumption \(\Gamma > 0\), the objective loss function is convex.

\textsuperscript{4}See Kim and Kim (2003) and Woodford (2011) for detailed discussions.

\textsuperscript{5}See Appendices C–F for detailed derivations.

\textsuperscript{6}Parameters \(q_A\), \(q_G\), and \(q_Z\) are defined in appendix F.
3.2 Linear Constraints

Constraints on the optimization problem come from log-linear approximations to the model equations. The first constraint comes from the aggregate supply relation between current inflation and output gap

\[ \hat{\pi}_t = \beta E_t[\hat{\pi}_{t+1}] + \kappa(\hat{x}_t + \psi\hat{\tau}_t) + u_t \]  

(21)

where \( u_t \) is a composite cost-push shock which depends on the three exogenous disturbances

\[ u_t = \kappa \left[ q_A - \frac{1 + \varphi}{\varphi + \sigma s_c^{-1}} \right] \hat{A}_t + \kappa \left[ q_G - \frac{\sigma s_g}{\varphi + \sigma s_c^{-1} s_c} \right] \hat{G}_t + \kappa q_Z \hat{Z}_t \]  

(22)

The exogenous disturbances produce cost-push effects through (22) because with a distorted steady state, they generate a time-varying gap between the flexible-price equilibrium level of output and the efficient level of output.

When \( \hat{\tau}_t \) is given exogenously, \( \kappa\psi\hat{\tau}_t + u_t \) prevents complete stabilization of inflation and welfare relevant output gap. Iterating forward on (21) yields

\[ \hat{\pi}_t = E_t \sum_{k=0}^{\infty} \beta^k \kappa \hat{x}_{t+k} + U_t \]

where \( U_t \equiv E_t \sum_{k=0}^{\infty} \beta^k (\kappa\psi\hat{\tau}_{t+k} + u_{t+k}) \) determines the degree to which stabilization of inflation and output gap is not possible. This is the only source of trade-off between stabilization of inflation and output gap in conventional optimal monetary policy analyses [for example, Galí (1991)].

When \( \hat{\tau}_t \) is chosen optimally along with monetary policy, then \( \hat{\tau}_t \) can be set to fully absorb cost-push disturbances, making simultaneous stabilization of inflation and the output gap possible. We rewrite (21), as in Benigno and Woodford (2004), in the form

\[ \hat{\pi}_t = \beta E_t[\hat{\pi}_{t+1}] + \kappa\hat{x}_t + \kappa\psi(\hat{\tau}_t - \hat{\tau}_t^*) \]  

(23)

where \( \hat{\tau}_t^* \equiv -\frac{1}{\kappa\psi}u_t \) is the tax rate that offsets the cost-push disturbances.

A second constraint arises from the household’s Euler equation. After imposing market clearing it may be written as

\[ \hat{x}_t = E_t[\hat{x}_{t+1}] - \frac{s_c}{\sigma} \left( \hat{\pi}_t - E_t[\hat{\pi}_{t+1}] \right) + v_t \]  

(24)

where the composite aggregate demand shock, \( v_t \), is

\[ v_t \equiv q_A(\rho_A - 1)\hat{A}_t + (q_G - s_g)(\rho_G - 1)\hat{G}_t + q_Z(\rho_Z - 1)\hat{Z}_t \]  

(25)
Alternatively, (24) can be written as

$$\hat{x}_t = E_t[\hat{x}_{t+1}] + \frac{s_c}{\sigma} E_t[\hat{\pi}_{t+1}] - \frac{s_c}{\sigma} (\hat{i}_t - \hat{i}_t^*)$$  \hspace{1cm} (26)

where $\hat{i}_t^* \equiv \sigma_s c \tau_t$ is the setting of the short-term nominal interest rate that exactly offsets composite demand-side disturbances.\(^7\)

Absence of arbitrage between short-term and long-term bonds delivers another constraint on the optimal policy program

$$\beta \rho E_t \hat{Q}_M t = \hat{Q}_t^M + \hat{i}_t$$  \hspace{1cm} (27)

Iterating on (27) and imposing a terminal condition yields

$$\hat{Q}_t^M = -E_t \sum_{k=0}^{\infty} (\beta \rho)^k \hat{i}_{t+k}$$  \hspace{1cm} (28)

Defining the long-term interest rate $i_t^M$ as the yield to maturity $\frac{1}{Q_t^M} - (1 - \rho)$, we obtain the term structure of interest rates

$$\hat{i}_t^M = \frac{1 - \beta \rho}{1 - \beta} E_t \sum_{k=0}^{\infty} (\beta \rho)^k \hat{i}_{t+k}$$  \hspace{1cm} (29)

When $\rho = 0$, so all bonds are one period, $\hat{i}_t^M = \frac{1}{1 - \beta} \hat{i}_t$, the long-term interest rate at time $t$ is proportional to the current short-term interest rate and any disturbance to the long rate will also affect the current short rate; when $\rho > 0$, the long-term interest rate at time $t$ is determined by the whole path of future short-term interest rates and a disturbance to the long-term interest rate can be absorbed by adjusting future short-term interest rates, with no change in the current short rate. That is, by separating current and future monetary policies, long bonds provide policy additional leverage.

The government’s budget constraint, constituting a fourth constraint, is

$$\hat{b}_{t-1}^M + f_t = \beta \hat{i}_t^M + (1 - \beta) \left( \frac{\bar{\pi}}{s_d} (\hat{\pi}_t + \hat{x}_t) + \hat{\pi}_t + \beta (1 - \rho) \hat{Q}_t^M \right)$$  \hspace{1cm} (30)

where $s_d \equiv \bar{S}/\bar{Y}$ is the steady state surplus-output ratio and $f_t$ is a composite shock that reflects all three exogenous disturbances to the government’s flow constraint

$$f_t = -(1 - \beta) \left( \frac{\bar{\pi}}{s_d} q_A \hat{A}_t + (1 - \beta) \left( \frac{s_g}{s_d} - \frac{\bar{\pi}}{s_d} q_G \right) \hat{G}_t + (1 - \beta) \left( \frac{s_z}{s_d} - \frac{\bar{\pi}}{s_d} q_Z \right) \hat{Z}_t \right)$$  \hspace{1cm} (31)

\(^7\)Note that $\hat{i}_t^* = \frac{\sigma_s}{s_d} E_t [(\hat{y}_{t+1}^* - \hat{y}_t^*) - s_g (\hat{G}_{t+1} - \hat{G}_t)]$, giving it an interpretation as the efficient level of the real interest rate.
In general, all disturbances will have fiscal consequences through (30) and (31) because nondistorting taxes are not available to offset their impacts on the government’s budget.

Iterating forward on (30), we obtain the intertemporal condition

$$\dot{b}_{t-1}^M + F_t = \dot{\pi}_t + \frac{\sigma}{s_c} \dot{x}_t + (1 - \beta) E_t \sum_{k=0}^{\infty} \beta^k [b_r(\dot{\tau}_{t+k} - \dot{\tau}^*_t) + b_x \hat{x}_{t+k}] - \rho \dot{Q}_t^M$$ (32)

where $b_r = \frac{\bar{\tau}_s}{s_d}$, $b_x = \frac{\bar{\tau}_s}{s_d} - \frac{\sigma}{s_c}$ and

$$F_t \equiv E_t \sum_{k=0}^{\infty} \beta^k f_{t+k} + (1 - \beta) b_r E_t \sum_{k=0}^{\infty} \beta^k \tau^*_{t+k} + E_t \sum_{k=0}^{\infty} \beta^{k+1} i^*_{t+k}$$ (33)

$F_t$ measures the extent to which exogenous “fiscal stress,” prevents the stabilization of inflation and output, as Benigno and Woodford (2007) note. Given the definitions of $\tau^*$ and $i^*$, $F_t$ reflects fiscal stress stemming from three conceptually distinct but related sources: a composite fiscal source, $f_t$, a composite cost-push source, $u_t$ (via $\tau^*_t$), and a composite aggregate demand source, $v_t$ (via $i^*_t$).

Long bonds can help to relieve the fiscal stress. Rewrite (32) to yield

$$\dot{b}_{t-1}^M + F'_t = \dot{\pi}_t + \frac{\sigma}{s_c} \dot{x}_t + (1 - \beta) E_t \sum_{k=0}^{\infty} \beta^k [b_r(\dot{\tau}_{t+k} - \dot{\tau}^*_t) + b_x \hat{x}_{t+k}]$$

$$+ E_t \sum_{k=0}^{\infty} (\beta \rho)^{k+1} (\dot{i}_{t+k} - \dot{i}^*_{t+k})$$ (34)

where

$$F'_t = F_t - E_t \sum_{k=0}^{\infty} (\beta \rho)^{k+1} i^*_{t+k}$$ (35)

The sum $\dot{b}_{t-1}^M + F'_t$ summarizes all the factors that prevent complete stabilization of inflation and the welfare-relevant output gap. Note that $F'_t \leq F_t$, with the wedge between them affected by the average maturity, $\rho$. When $\rho = 0$ and only one-period bonds exist, $F'_t$ reduces to $F_t$. At the opposite extreme, it all bonds are consols, $\rho = 1$, $F'_t$ no longer depends on $i^*_t$, the composite aggregate demand source of disturbance.

4 Optimal Policy Analytics: Flexible Prices

The special case of completely flexible prices clearly illustrates the role of long-term nominal bonds in the optimal policy and it connects to earlier work by Chari, Christiano, and Kehoe
Flexible prices emerge when $\theta = 0$, which implies $\kappa = \infty$ and $q_* = 0$, so there is no longer a tradeoff between output and inflation. Costless inflation converts the loss function from (20) to

$$\frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t q_* \hat{x}_t^2 \quad (36)$$

The optimal policy problem minimizes (36) subject to the sequence of constraints

$$\dot{x}_t + \psi(\hat{\tau}_t - \hat{\tau}_t^*) = 0 \quad (37)$$

$$\dot{x}_t = E_t[\hat{x}_{t+1}] + \frac{s_r}{s_c} E_t[\hat{\pi}_{t+1}] - \frac{s_r}{\sigma} (\hat{i}_t - \hat{i}_t^*) \quad (38)$$

$$\dot{b}_M = F_t = \hat{\pi}_t + \frac{\sigma}{s_c} \hat{x}_t + (1 - \beta) E_t \sum_{k=0}^{\infty} \beta^k [\beta_r (\hat{\tau}_{t+k} - \hat{\tau}_{t+k}^*) + b_x \hat{x}_{t+k}]$$

$$+ E_t \sum_{k=0}^{\infty} (\beta \rho)^{k+1} (\hat{i}_{t+k} - \hat{i}_{t+k}^*) \quad (39)$$

The optimal solution entails $\dot{x}_t = 0$ at all times, which can be achieved if fiscal policy follows $\hat{\tau}_t = \hat{\tau}_t^*$ and monetary policy sets the short-term real interest rate as $\hat{i}_t - E_t \hat{\pi}_{t+1} = \hat{i}_t^*$. Equilibrium inflation satisfies

$$\dot{b}_M + F_t = \hat{\pi}_t + E_t \sum_{k=1}^{\infty} (\beta \rho)^k \hat{\pi}_{t+k} \quad (40)$$

so increases in factors that prevent complete stabilization of the objectives, raise the expected present value of inflation. When $\rho > 0$, (40) implies that long-term bonds allow the government to trade off inflation today for inflation in the future. The longer the average maturity, the farther into the future inflation can be postponed. This conclusion is reminiscent of Cochrane’s (2001) optimal inflation-smoothing result.

When $\rho = 0$ and all bonds are one-period, (40) collapses to

$$\dot{b}_M + F_t = \hat{\pi}_t \quad (41)$$

and, as Benigno and Woodford (2007) emphasize, “optimal policy will involve highly volatile inflation and extreme sensitivity of inflation to fiscal shocks.”

Flexible prices neglect the welfare costs of variations in inflation. When prices are sticky and inflation volatility is costly, the optimal allocation should balance between variations in inflation and variations in the output gap.
5 Optimal Policy Analytics: Sticky Prices

In the case where prices are sticky, the optimization problem finds paths for \( \{\hat{\pi}_t, \hat{x}_t, \hat{\tau}_t, \hat{i}_t, \hat{b}_M^t, \hat{Q}_M^t\} \) that minimize

\[
\frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t [\hat{\pi}_t^2 + \lambda \hat{x}_t^2], \quad \lambda \equiv \frac{q_x}{q_\pi}
\]  

subject to the sequence of constraints

\[
\hat{\pi}_t = \beta E_t [\hat{\pi}_{t+1}] + \kappa \hat{x}_t + \kappa \psi (\hat{\tau}_t - \hat{\tau}_t^*) \quad (43)
\]
\[
\hat{x}_t = E_t [\hat{x}_{t+1}] + \frac{s_c}{\sigma} E_t [\hat{\pi}_{t+1}] - \frac{s_c}{\sigma} (\hat{\hat{\tau}}_t - \hat{\tau}_t^*) \quad (44)
\]
\[
\hat{b}_M^{t-1} = \beta \hat{b}_t^M + (1 - \beta) \frac{\bar{\tau}}{s_d} (\hat{\tau}_t + \hat{x}_t) + \hat{\pi}_t + \beta (1 - \rho) \hat{Q}_t^M - f_t \quad (45)
\]
\[
\hat{Q}_t^M = \beta \rho E_t \hat{Q}_{t+1}^M - \hat{i}_t \quad (46)
\]

First-order conditions are

\[
\hat{\pi}_t - L_t^\pi + L_{t-1}^\pi + \frac{s_c}{\sigma \beta} L_{t-1}^x + L_t^b = 0 \quad (47)
\]
\[
\lambda \hat{x}_t + \kappa L_t^\pi - L_t^x + \frac{1}{\beta} L_{t-1}^x + (1 - \beta) \frac{\bar{\tau}}{s_d} L_t^b = 0 \quad (48)
\]
\[
\frac{s_c}{\sigma} L_t^x + L_t^q = 0 \quad (49)
\]
\[
\kappa \psi L_t^\pi + (1 - \beta) \frac{\bar{\tau}}{s_d} L_t^b = 0 \quad (50)
\]
\[
\beta (1 - \rho) L_t^b - L_t^q + \rho L_{t-1}^q = 0 \quad (51)
\]
\[
E_t L_{t+1}^b - L_t^b = 0 \quad (52)
\]

where \( L_t^\pi, L_t^x, L_t^b, L_t^q \) are Lagrange multipliers corresponding to (43)–(46). We solve (43)–(52) for state-contingent paths of \( \{\hat{\pi}_t, \hat{x}_t, \hat{\tau}_t, \hat{i}_t, \hat{b}_M^t, \hat{Q}_M^t, L_t^\pi, L_t^x, L_t^b, L_t^q\} \).

From (49) and (50)

\[
L_t^x = - \frac{\sigma}{s_c} L_t^q \quad (53)
\]
\[
L_t^\pi = - \frac{1 - \beta}{\kappa \psi \bar{\tau}} L_t^b \quad (54)
\]
Substitute these into (47) and (48) to yield

\[
\hat{\pi}_t = -\frac{1 - \beta}{\kappa \psi} \bar{s}_d (L^b_t - L^b_{t-1}) - L^b_t + \frac{1}{\beta} L^q_{t-1}
\]

(55)

\[
\lambda \hat{e}_t = (\psi^{-1} - 1)(1 - \beta) \frac{\bar{r}}{s_d} L^b_t - \frac{\sigma}{s_c} L^q_t + \frac{\sigma}{s_c \beta} L^q_{t-1}
\]

(56)

With inflation and the output gap expressed as functions of only \(L^b_t\) and \(L^q_t\), it is clear that any disturbance that affects the government budget or debt maturity constraints impacts inflation and output. First-order condition (51) links \(L^q_t\) to a distributed lag of \(L^b_t\) with weights that decay with \(\rho\), the determinant of debt’s duration

\[
L^q_t = \beta (1 - \rho) \sum_{k=0}^{\infty} \rho^k L^b_{t-k}
\]

(57)

Evidently, maturity structure matters through its implications for fiscal financing. Inflation and output-gap fluctuations depend on the entire history of multipliers on the government budget, \(L^b_{t-j}\), and the degree of history dependence rises with the average maturity of government debt. Restricting attention to only one-period debt, so \(\rho = 0\), eliminates the history dependence.\(^8\) At the opposite extreme, considering only consols, so \(\rho = 1\), ensures \(L^q_t \equiv 0\), regardless of how binding the government’s budget has been in the past.\(^9\)

More generally, the price of long bonds can adjust to relax how binding the government’s budget constraint may be. And the term structure relation, (29), connects the price of bonds today to future short-term interest rates. Debt maturity introduces a fresh role for expected monetary policy choices by allowing those expectations to help ensure government solvency.\(^10\)

We examine some special cases that allow us to characterize the optimal equilibrium analytically.

---

\(^8\)This is precisely the exercise that finds the combination of active monetary/passive fiscal policies yields highest welfare [Schmitt-Grohé and Uribe (2007) and Kirsanova and Wren-Lewis (2012)].

\(^9\)Sims (2013) limits attention to this case.

\(^10\)The new Keynesian literature emphasizes the role of expected monetary policy via its influence of the entire future path of \textit{ex-ante} real interest rates that enter the Euler equation, (24). The role we are discussing for expected monetary policy is in addition to this conventional role.
5.1 Only One-Period Bonds  Suppose the government issues only one-period bonds, rolled over every period. Then \( \rho = 0 \) and (57) and (29) reduce to

\[ L_t^g = \beta L_t^b \tag{58} \]
\[ \hat{\gamma}_t^M = \frac{1}{1 - \beta} \hat{\gamma}_t \tag{59} \]

Long-term and short-term interest rates are identical, so \( L_t^b \) and \( L_t^g \) covary perfectly. In this case, (55) and (56) become

\[ \hat{\pi}_t = - \left( \frac{1 - \beta}{\kappa \psi} \frac{\bar{r}}{s_d} + 1 \right) (L_t^b - L_{t-1}^b) \tag{60} \]
\[ \lambda \hat{x}_t = \left[ (\psi^{-1} - 1) (1 - \beta) \frac{\bar{r}}{s_d} - \beta \frac{\sigma}{s_c} \right] L_t^b + \frac{\sigma}{s_c} L_{t-1}^b \tag{61} \]

Condition (60) implies that inflation is proportional to the forecast error in \( L_t^b \). Because (52) requires there are no forecastable variations in \( L_t^b \), the expectation of inflation is zero:

\[ E_t \hat{\pi}_{t+1} = 0 \quad \Rightarrow \quad E_t \hat{\pi}_{t+1} = \hat{\pi}_t \tag{62} \]

It is instructive to link the result in (62) to the government solvency condition

\[ \hat{b}_t^M + \beta \rho \hat{Q}_t^M - \hat{\pi}_t = (1 - \beta) E_t \sum_{k=0}^{\infty} \beta^k (\hat{r}_{t,t+k} + \hat{s}_{t+k}) \tag{63} \]

where \( \hat{r}_{t,t+k} \) is the log-linearized real stochastic discount factor, \( \beta \frac{U_{t,t+k}}{U_{t,c.t}} \). When there are no long-term bonds, \( \rho = 0 \) so the bond price disappears and any news about the expected present value of surpluses must be absorbed by current inflation \( \hat{\pi}_t \).

Condition (61) makes the output gap a weighted average of \( L_t^b \) and \( L_{t-1}^b \). Taking expectations yields

\[ \lambda E_t \hat{x}_{t+1} = \left[ (\psi^{-1} - 1) (1 - \beta) \frac{\bar{r}}{s_d} + (1 - \beta) \frac{\sigma}{s_c} \right] L_t^b \tag{64} \]

which makes clear how solvency considerations affect expectations. Combining this with (61) yields

\[ \lambda (E_t \hat{x}_{t+1} - x_t) = \frac{\sigma}{s_c} (L_t^b - L_{t-1}^b) \tag{65} \]

so the expected change in the output gap next period is proportional to the surprise in the multiplier on government solvency today. The optimal degree of output-gap smoothing

\[ \text{First-order condition (52) makes } E_t L_{t+1}^b = L_t^b, \text{ so the surprise is } L_{t+1}^b - E_t L_{t+1}^b = \Delta L_{t+1}^b. \]
varies inversely with $\lambda$, the weight on output in the loss function. Under most calibrations, $\lambda$ is quite small, implying little smoothing of output. Taking expectations of (65), we obtain

$$E_{t-1}(\hat{x}_{t+1} - \hat{x}_t) = 0$$

(66)

We have expressed inflation and the output gap, and expectations of them, in terms of $L^b_t$ and $L^b_{t-1}$. Substituting into (43) and (44), we obtain

$$\hat{\tau}_t - \hat{\tau}^*_t = \frac{1}{\kappa \psi}(\hat{\pi}_t - \kappa \hat{x}_t)$$

$$= -\left\{ \frac{1}{\kappa \psi} \left( \frac{1 - \beta}{\kappa \psi} s_d + 1 \right) + \frac{1}{\lambda \psi} \left( (\psi^{-1} - 1) \frac{\hat{\tau}}{s_d} - \frac{\beta \sigma}{s_c} \right) \right\} L^b_t$$

$$+ \left[ \frac{1}{\kappa \psi} \left( \frac{1 - \beta}{\kappa \psi} \frac{\hat{\tau}}{s_d} + 1 \right) - \frac{1}{\lambda \psi} \frac{\sigma}{s_c} \right] L^b_{t-1}$$

(67)

$$\hat{i}_t - \hat{i}^*_t = \frac{\sigma}{s_c} (E_t \hat{x}_{t+1} - \hat{x}_t) = \left( \frac{\sigma}{s_c} \right)^2 \frac{1}{\lambda} (L^b_t - L^b_{t-1})$$

(68)

Each variable can now be expressed as a function of a single variable $L^b_t$. Substituting (67) and (68) into the government solvency condition, (34), yields the law of motion for $L^b_t$

$$L^b_t = \frac{m_b}{m_b + n_b} L^b_{t-1} - \frac{1}{m_b + n_b} [F' + \hat{b}^M_{t-1}]$$

(69)

Condition (69) implies that temporary disturbances in fiscal stress produce permanent changes in the multiplier $L^b_t$ and, hence, permanent changes in the output gap and the price level. Because both the price level $\hat{p}_t$ and the multiplier $L^b_t$ are random walks, smoothing the shadow value of the government budget smooths the price level.

5.2 Only Consols

Suppose the government issues only consols. With $\rho = 1$, (57) and (29) reduce to

$$L^q_t = L^q_{t-1} = 0$$

(70)

$$\hat{i}^M_t = E_t \sum_{k=0}^{\infty} \beta^k \hat{i}_{t+k}$$

(71)

In the case of consols, the long-term interest rate is determined by the entire path of future short-term interest rates. Fiscal stress that moves long rates need not change short rates contemporaneous, so long as the expected path of short rates satisfies (71). Inflation

---

12The coefficients in (69) are defined in appendix G.
and output are now
\[
\hat{\pi}_t = -\frac{1 - \beta}{\kappa \psi} \bar{\tau} (L_t^b - I_{t-1}^b) - L_t^b
\] (72)
\[
\lambda \hat{x}_t = \left(\psi^{-1} - 1\right) \left(1 - \beta\right) \bar{\tau} s_d L_t^b
\] (73)

Combining (72) and (73)
\[
\hat{\pi}_t + \frac{\lambda}{\kappa (1 - \psi)} (\hat{x}_t - \hat{x}_{t-1}) + \frac{\lambda}{(\psi^{-1} - 1)(1 - \beta) s_d} \hat{x}_t = 0
\] (74)

an expression that generalizes the “flexible target criterion” found in conventional optimal monetary policy exercises in new Keynesian models.\(^{13}\)

Condition (73) makes the output gap proportional to \(L_t^b\). The first-order condition for debt, (52), implies there are no forecastable variations in \(L_t^b\), so there are no forecastable variations in \(x_t\). Taking expectations of (73), we have
\[
E_t \hat{x}_{t+1} = \hat{x}_t
\] (75)

It is interesting to notice that with long-term debt, the output gap is proportional to the Lagrange multiplier associated with government budget. Since \(L_t^b\) is random walk, the output gap is also a random walk. To understand this, it is useful to refer back to the government solvency condition
\[
\hat{b}_{t-1}^M + \beta p \hat{Q}_t^M - \hat{\pi}_t = (1 - \beta) E_t \sum_{k=0}^{\infty} \beta^k (\hat{r}_{t+k} + \hat{s}_{t+k})
\] (63)

Consols introduce the possibility that the bond price \(\hat{Q}_t^M\) can behave as a fiscal shock absorber: bad news about future surpluses can reduce the value of outstanding bonds, leaving the real discount rate unaffected. A constant real discount rate smoothes the output gap, which explains the absence of forecastable variations in the output gap. Higher is the value of \(\rho\)—longer the duration of debt—the less is the required change in bond prices and future inflation for a given change in present-value surpluses.

Condition (72) implies that inflation is determined by a weighted average of \(L_t^b\) and \(L_{t-1}^b\).

---

\(^{13}\)Notice that as \(\psi \to 0\), which occurs as the steady state distorting tax rate approaches 0, \(L_t^b \to 0\) and (74) approaches the conventional flexible target criterion with lump-sum taxes \(\hat{\pi}_t + \frac{1}{\kappa \psi} (\hat{x}_t - \hat{x}_{t-1}) = 0\) so that optimal inflation rate should vary with both the the rate of change in the output gap and the level of the gap [see Woodford (2011) and references therein].
Taking expectations

\[ E_t \hat{\pi}_{t+1} = -L_t^b = -\frac{\lambda}{(1/\psi - 1)(1 - \beta)} \frac{L_t^b}{s_d} \hat{x}_t \]  

(76)

or

\[ E_t \hat{\pi}_{t+1} = \hat{\pi}_t + \frac{1 - \beta \bar{\tau}}{\kappa \psi} (L_t^b - L_{t-1}^b) \]  

(77)

(77) implies that the expectation of inflation next period is formed by current period’s realized inflation plus an adjustment term that is proportional to the forecasting error of \( L_t^b \).

Expectations of this expression conditional on information at \( t - 1 \) yields

\[ E_{t-1}(\hat{\pi}_{t+1} - \hat{\pi}_t) = 0 \]  

(78)

Substituting the above results into (43) and (44) yields tax-rate and interest-rate gaps

\[ \hat{\tau}_t - \hat{\tau}_t^* = \frac{1}{\kappa \psi}(\hat{\pi}_t - \beta E_t \hat{x}_{t+1} - \kappa \hat{x}_t) \]

\[ = - \left[ \frac{1}{\kappa \psi} \left( \frac{1 - \beta \bar{\tau}}{\kappa \psi} \frac{1 - \beta}{s_d} \right) + \frac{1}{\psi \lambda} \frac{1}{\psi} (1 - \beta) \bar{\tau} \right] L_t^b + \frac{1}{\kappa \psi} \frac{1 - \beta \bar{\tau}}{s_d} L_{t-1}^b \]  

(79)

\[ \hat{i}_t - \hat{i}_t^* = \frac{\sigma}{s_c} E_t \hat{\pi}_{t+1} = -L_t^b \]  

(80)

Substituting into the government’s intertemporal condition gives

\[ L_t^b = \frac{\bar{m}_b}{\bar{m}_b + \bar{n}_b} L_{t-1}^b - \frac{1}{\bar{m}_b + \bar{n}_b} [F_t + \hat{b}_{t-1}^M] \]  

(81)

\[ \hat{b}_t^M = -E_t F_{t+1} - \bar{n}_b L_t^b \]  

(82)

where the coefficients in these expressions are defined in appendix G.

5.3 General Case  
Rewrite (55) and (56) using the lag-operator notation, \( \mathbb{L} x_t \equiv x_{t-j} \)

\[ \hat{\pi}_t = -\frac{(1 - \beta) \bar{\tau}}{\kappa \psi} s_d (1 - \mathbb{L}) L_t^b - (1 - \mathbb{L})(1 - \rho \mathbb{L})^{-1} L_t^b \]  

(83)

\[ \lambda \hat{x}_t = (\psi^{-1} - 1)(1 - \beta) \frac{s_d}{s_d} L_t^b - \frac{\sigma \beta}{s_c} (1 - \rho)(1 - \beta^{-1} \mathbb{L})(1 - \rho \mathbb{L})^{-1} L_t^b \]  

(84)

The optimality condition for debt that requires \( L_t^b \) to be a martingale may be written as

\[ (1 - \mathbb{B}) E_{t-1} L_t^b = 0 \]  

(85)
where $B$ is the backshift operator, defined as $B^{-j} E_t \xi_t \equiv E_t \xi_{t+j}$.

Taking expectations of (83) and (84), we obtain

$$E_t \hat{\pi}_{t+1} = -\frac{(1 - \beta) \bar{\psi}}{\kappa \psi} (1 - B) E_t L^b_{t+1} - (1 - B) (1 - \rho B)^{-1} E_t L^b_{t+1}$$  \hspace{1cm} (86)$$

$$\lambda E_t \hat{x}_{t+1} = (\psi^{-1} - 1) (1 - \beta) \frac{\bar{\pi}}{s_d} E_t L^b_{t+1} - s_c^{-1} \sigma (1 - \rho) (1 - \beta^{-1} B)(1 - \rho B)^{-1} E_t L^b_{t+1}$$  \hspace{1cm} (87)$$

Applying (85), we obtain

$$E_t \hat{\pi}_{t+1} = \rho \hat{\pi}_t + \frac{1 - \beta}{s_d} \frac{\bar{\pi}}{\kappa \psi} (L^b_t - L^b_{t-1})$$  \hspace{1cm} (88)$$

$$E_t \hat{x}_{t+1} = \rho \hat{x}_t + (1 - \rho) \frac{1 - \beta}{\lambda} \frac{\sigma}{s_c} L^b_t$$  \hspace{1cm} (89)$$

(88) and (89) are important to understand the effects of maturity structure on the formation of expectations. The expectation of inflation and output-gap are both history dependent. The expectation of next period’s inflation depends on the realization of inflation this period plus an adjustment term, which is proportional to the forecasting error of $L^b_t$. Taking expectations at time $t - 1$ yields

$$E_{t-1} \hat{\pi}_{t+1} = \rho E_{t-1} \hat{\pi}_t$$  \hspace{1cm} (90)$$

Expression (90) implies that the expectation of future inflation decays at rate $\rho$. The longer the duration, the more persistent is expected future inflation. The expectation of the output gap also depends on a linear combination of the realization of the current output gap and an adjustment term. The longer the average maturity, the more weight households put on the lagged output gap and the less weight on the current Lagrange multiplier. Longer average maturity makes the output gap less susceptible to current disturbances in fiscal stress.

6 Numerical Results

We turn to numerical results from the model calibrated to U.S. data in order to focus on a set of implications that may apply to an actual economy.

Table 2 reports a calibration to U.S. time series. We take the model’s frequency to be quarterly and adopt some parameter values from Benigno and Woodford (2004), including $\beta = 0.99$, $\theta = 0.66$ and $\epsilon = 10$; we set $\varphi = \sigma = 0.5$, implying a Frisch elasticity and an intertemporal elasticity of substitution of 2.0, both reasonable empirical values. Quarterly U.S. data from 1948Q1 to 2013Q1 underlie the values of $s_b$, $s_g$, $s_z$ and are used to estimate
autoregressive processes for the three exogenous shocks.\footnote{Appendix H provides details.} Table 2’s calibration makes the relative weight on output-gap stabilization equal to $\lambda = 0.0033$, slightly higher than the value used in Benigno and Woodford (2007) ($\lambda = 0.0024$).\footnote{Benigno and Woodford’s calibration of $\sigma = 0.16$ largely explains the difference in the values of $\lambda$.}

<table>
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<tr>
<th>parameter</th>
<th>definition</th>
<th>value</th>
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<td>$\beta$</td>
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<td>$\sigma$</td>
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</tr>
<tr>
<td>$\sigma_z^z$</td>
<td>standard deviation of innovation to transfer payment shock</td>
<td>0.047</td>
</tr>
</tbody>
</table>

Table 2: Calibration to U.S. Data

6.1 Fiscal Financing  Sims (2013) emphasizes the role of surprise inflation as a “fiscal cushion” that can reduce the reliance on distorting sources of revenues. One way to quantify the fiscal cushion is to use the government’s solvency condition to account for the sources of fiscal financing—including current and future inflation—following an innovation in the present value of transfers, $\hat{Z}$, or government purchases, $\hat{G}$. The solvency condition may be
written as
\[
(1 - \beta) E_t \sum_{k=0}^{\infty} \beta^k \left[ \frac{s_z}{s_d} \hat{Z}_{t+k} + \frac{s_g}{s_d} \hat{C}_{t+k} \right] = -\dot{b}_t^M + \hat{\pi}_t \quad \text{PV(government expenditures)}
\]
\[
+ E_t \sum_{k=1}^{\infty} (\beta \rho)^k \hat{\pi}_{t+k} - E_t \sum_{k=1}^{\infty} \left[ (1 - \beta) \beta^k - (1 - \beta \rho)(\beta \rho)^k \right] \sigma(\hat{c}_{t+k} - \hat{c}_t)
\quad \text{PV(future inflation)}
\]
\[
E_t \sum_{k=0}^{\infty} \beta^k (\hat{\pi}_{t+k} + \hat{y}_{t+k})
\quad \text{PV(tax revenues)}
\]
\[
E_t \sum_{k=0}^{\infty} \beta^k (\hat{\tau}_{t+k} + \hat{v}_{t+k})
\quad \text{PV(future inflation)}
\]
\[
(1 - \beta \rho) (\beta \rho)^k \sigma(\hat{c}_{t+k} - \hat{c}_t)
\quad \text{PV(real interest rate)}
\]
\[
(91)
\]

Condition (91) states that a fiscal disturbance that changes the present value of government expenditures must be financed by some mix of current and future tax revenues, current and future inflation rates—which revalue outstanding government bonds—and future real discount rates—which alter the present value of the revenue flows. Fiscal financing underscores the inherent symmetry between monetary and fiscal policy: interactions between the two policies determine the reliance on tax revenues versus current and future inflation.

Figure 2 plots the financing decomposition for a government transfers shock—top panels—and a government purchases shock—bottom panels—as a function of the average duration of government bonds for the calibration to U.S. data in Table 2. For both transfers and purchases shocks, the vast majority of financing comes from future tax revenues—in fact, for government purchases, it is nearly 100 percent tax-financed, regardless of the duration of government debt. Both current and future inflation are relatively unimportant sources of financing, though future inflation becomes relatively more important as the duration of debt increases because changes in bond prices can help to maintain government solvency.

As the average duration increases, \( \rho \to 1 \), (91) and the upper right panel of the figure both show that the importance of real interest rate adjustments dissipates. In the new Keynesian model, real interest rates transmit immediately into movements in the output gap, so at short durations, distortions in output are relatively big. As duration rises, it is optimal to smooth output more, so real interest rate movements diminish. In the limit, when \( \rho = 1 \), the present value of real interest rate is zero and it is optimal to make \( E_t \hat{x}_{t+1} = \hat{x}_t \) and rely instead on inflation as a fiscal cushion.

Source of fiscal financing are particularly sensitive to the level of debt in the economy. Figure 3 reports fiscal financing of an increase in transfers under three steady state debt-GDP levels: the calibration to U.S. data (49 percent), “low debt” (20 percent), “high debt” (100 percent). Figure 4 repeats the analysis for an increase in government purchases. As the level

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16 More generally, spending increases could be financed by future spending decreases, but those adjustments are precluded by assumption in the optimal policy exercises we perform.
of debt rises, the reliance on tax financing declines. With very short debt duration, changes in real interest rates account for a substantial fraction of financing in high-debt economies. Reliance on real rates declines rapidly as duration rises, with future inflation becoming increasingly important. With long-duration debt, high-debt economies would finance over 15 percent of a transfers innovation with current and future inflation.

The roles of current and future inflation hold up for the financing of an increase in government purchases. Real interest rates, whose movements go against financing higher government purchases, account for about seven percent of the financing at short durations. As with transfers, reliance on tax revenues declines as duration increases.

6.2 Dynamic Impacts of Disturbances Figures 5–7 plot the dynamic responses to shocks in government purchases, government transfers, and total factor productivity under three different debt durations—one quarter (solid lines), five years (dashed lines) and a consol (dotted-dashed lines).

6.3 Contrast to Conventional Optimal Monetary Policy The conventional optimal monetary policy problem, as Woodford (2011) describes, typically assumes a nondistorting source of revenue exists, so that stabilization policy abstracts from fiscal policy dis-
Figure 3: Fiscal financing of exogenous increase in transfers in percentage of innovation in expenditures. Solid line: U.S. debt-GDP ratio (49%); dashed line: low debt (20%); dotted-dashed line: high debt (100%).

Figure 4: Fiscal financing of exogenous increase in government purchases in percentage of innovation in expenditures. Solid line: U.S. debt-GDP ratio (49%); dashed line: low debt (20%); dotted-dashed line: high debt (100%).
Figure 5: Exogenous increase in government spending

Figure 6: Exogenous increase in government transfers
Figure 7: Exogenous increase in total factor productivity

tortions. To place the conventional optimal problem on an equal footing with the fully optimal problem, we have the government optimally choose the interest rate function, taking as given exogenous processes for technology, government spending, and the distorting tax rate; lump-sum transfers (or taxes) adjust passively to ensure the government’s solvency condition never binds. In the conventional problem, the maturity structure of debt is irrelevant. In this section we contrast fully optimal policy to the conventional optimal monetary policy. In both cases, we examine the case of a distorted steady state. We focus on optimal policy commitment and adopt Woodford’s (2003) “timeless perspective.”

Figure 8 plots the value of the loss function under the conventional optimal monetary policy—highest dashed line—and under the fully optimal monetary and fiscal policies—solid line—as a function of the average duration of government debt. These calculations employ the calibration in table 2 for U.S. data. Even though fully optimal policies do not use lump-sum taxes to make the government solvency condition non-binding, welfare is higher under fully optimal policies. The reason for higher welfare is that in U.S. data, average tax rates are persistent ($\rho_\tau = 0.782$) and volatile ($\sigma_\tau^e = 0.029$) and welfare under conventional optimal monetary policy suffers from variable tax rates. The figure shows that

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17Key earlier expositions of the conventional optimal monetary policy problem include Clarida, Galí, and Gertler (1999) and Woodford (2003).
by reducing the volatility of the tax process—rescaling it to be comparable to the government spending or the productivity processes—welfare under the two optimal policy regimes can be made equivalent. Of course, because the maturity of government debt is irrelevant under conventional optimal monetary policy, as maturity changes, there is nothing that policy regime can do to adjust.

Figure 8: Solid line is value of loss function under fully optimal policies; dashed lines are loss function under conventional optimal monetary policy with passive lump-sum taxes. Welfare can be equivalent between the two policies by scaling the variance of the distorting tax appropriately.

Figure 9 reports the implications of debt levels for the increase in welfare under fully optimal policies compared to conventional optimal monetary policy. For all debt-GDP ratios, welfare increases monotonically with average duration under the fully optimal policies, but not under the conventional optimal monetary policy. Welfare improvements are most dramatic at the short end of the maturity structure—up to about five years—and most pronounced in high-debt economies. This suggests an important policy message: high-debt economies may have difficulty placing long-maturity debt, but they may derive substantial benefits from selling even moderate maturity bonds.
Figure 9: Increase in welfare under fully optimal policies compared to conventional optimal monetary policy with passive lump-sum taxes, in units of fraction of steady state output. Solid line: U.S. debt-GDP ratio (49%); dashed line: low debt (20%); dotted-dashed line: high debt (100%).

7 Concluding Remarks

This paper shows that the maturity structure of outstanding government debt can play an important role in stabilization policies. Optimal monetary and fiscal policies and the resulting equilibria depend explicitly on the average maturity of government debt.

The paper also shows that revaluations of nominal government debt through surprise changes in current and expected inflation are components of an optimal monetary-fiscal policy regime, expanding on a suggestion that Sims (2013) makes. Surprise inflation allows the government to rely less on distorting taxes to ensure government solvency. Longer maturity government bonds permit the inflation to be smoothed over time, reducing the welfare losses that would otherwise occur due to the price dispersion that inflation creates.

Inflation’s role as a fiscal shock absorber grows more important as the average level of government debt rises in an economy. Once debt grows to about 100 percent of GDP, an optimal monetary-fiscal policy mix can entail current and future inflation accounting for 15 percent or more of the financing of government spending or transfers disturbances. This optimal policy contrasts sharply with the fiscal financing in inflation-targeting policy regimes, where all fiscal disturbances are financed by future taxes.
REFERENCES


A Derivation of Long-term Bond Price and IEC

Define

\[ Q_{t,t+k} = \beta^k \frac{U_{c,t+k}}{U_{c,t}} \frac{P_t}{P_{t+k}} \]  

(A.1)

as the stochastic discount factor for the price at \( t \) of one unit of composite consumption goods at \( t + k \). Then (3) and (4) in the text can be written as

\[ Q^S_t = E_t Q_{t,t+1} \]  

(A.2)

\[ Q^M_t = E_t Q_{t,t+1}(1 + \rho Q^M_{t+1}) \]  

(A.3)

Iterating on (A.3) and imposing a terminal condition yields

\[ Q^M_t = E_t [Q_{t,t+1} + \rho Q_{t,t+1} Q^M_{t+1}] \]

\[ = E_t \{Q_{t,t+1} + \rho Q_{t,t+1} E_{t+1} Q_{t+1,t+2} + \rho^2 Q_{t,t+1} E_{t+1} Q_{t+1,t+2} + \rho^3 E_{t+1} Q_{t+1,t+2} Q^S_{t+2} + \ldots \} \]

\[ = Q^S_t + \rho E_t [Q_{t,t+1} Q^S_{t+1}] + \rho^2 E_t [Q_{t,t+1} E_{t+1} Q^S_{t+2}] + \rho^3 E_t [Q_{t,t+2} Q^S_{t+2}] + \ldots + \rho^k E_t [Q_{t,t+k} Q^S_{t+k}] + \ldots \]

\[ = Q^S_t + E_t \sum_{k=1}^{\infty} \rho^k E_t [Q_{t,t+k} Q^S_{t+k}] \]  

(A.4)

Equation (A.4) implies that the long-term bond’s price is determined by weighted average of expectations of future short-term bond’s prices.

Substitute (A.1) into (A.4)

\[ Q^M_t = E_t \left[ \frac{\beta U_{c,t+1}}{U_{c,t}} \frac{P_t}{P_{t+1}} + \rho \frac{\beta^2 U_{c,t+2}}{U_{c,t}} \frac{P_t}{P_{t+2}} + \ldots + \rho^{k-1} \beta^k \frac{U_{c,t+k}}{U_{c,t}} \frac{P_t}{P_{t+k}} \right] \]

\[ = E_t \left[ \sum_{k=1}^{\infty} \rho^{k-1} \beta^k \frac{U_{c,t+k}}{U_{c,t}} \frac{P_t}{P_{t+k}} \right] \]  

(A.5)

Condition (A.5) implies that the long-term bond price is determined by the whole path of expected future price level, discounted by consumption growth rate. The long-term bond price is negatively correlated with expected future inflation rate and consumption growth rate.

Rewrite (A.3) as

\[ Q^M_t = E_t Q_{t,t+1}(1 + \rho Q^M_{t+1}) \]

\[ = E_t Q_{t,t+1} E_t(1 + \rho Q^M_{t+1}) + \rho \text{cov}(Q_{t,t+1}, Q^M_{t+1}) \]

\[ = E_t Q^S_t(1 + \rho Q^M_{t+1}) + \rho \text{cov}(Q_{t,t+1}, Q^M_{t+1}) \]  

(A.6)
Recall from (A.5) that

\[ Q_{t+1}^M = E_t \sum_{k=1}^{\infty} \rho^{k-1} Q_{t+1, t+1+k} \]

\( Q_{t+1}^M \) is determined by weighted average of expected future discounted value of future stochastic discount factors. Therefore, without loss of generality, we assume \( \text{cov}(Q_{t,t+1}, Q_{t+1}^M) = 0 \), and (A.6) can be expressed as

\[ Q_t^M = E_t Q_t^S (1 + \rho Q_{t+1}^M) \quad (A.7) \]

To derive intertemporal equilibrium condition, we iterate on government’s period budget constraint, (13), and impose asset-pricing relations and the household’s transversality condition:

\[
(1 + \rho Q_t^M) \frac{B_{t-1}^M}{P_t} = Q_t^M \frac{B_t^M}{P_t} + S_t
\]

\[
= E_t \left[ \frac{Q_t^M \pi_{t+1}}{1 + \rho Q_{t+1}^M} \frac{B_t^M}{P_t} \right] + \frac{Q_t^M \pi_{t+1}}{1 + \rho Q_{t+1}^M} S_{t+1} + S_t
\]

\[
= E_t [S_t + \frac{Q_t^M \pi_{t+1}}{1 + \rho Q_{t+1}^M} S_{t+1} + \frac{Q_t^M \pi_{t+1} \pi_{t+2}}{(1 + \rho Q_{t+1}^M)(1 + \rho Q_{t+2}^M)} S_{t+2} + \ldots] \quad (A.8)
\]

Substituting (A.1) and (A.3) into (A.8) yields

\[
(1 + \rho Q_t^M) \frac{B_{t-1}^M}{P_t} = E_t \sum_{i=0}^{\infty} \beta^i \frac{U_{c,t+i}}{U_{c,t}} S_{t+i} \quad (A.9)
\]

To derive (15) in the text, combine (A.5) and (A.9)

\[
E_t \left( \sum_{k=0}^{\infty} \rho^k \beta^k \left( \frac{U_{c,t+k}}{U_{c,t}} \right) \frac{1}{P_{t+k}} \right) B_{t-1}^M = E_t \left( \sum_{i=0}^{\infty} \beta^i \frac{U_{c,t+i}}{U_{c,t}} \right) S_{t+i} \quad (A.10)
\]

**B Derivation of Nonlinear First Order Conditions**

The full nonlinear optimal policy problem maximizes

\[
E_0 \sum_{t=0}^{\infty} \beta^t U(Y_t, \Delta_t, \xi_t)
\]
subject to
\[
\left[\frac{1 - \theta \pi_t^{e^{-1}}}{1 - \epsilon}\right]^{1+\varphi} - \frac{1}{1 - \epsilon} \frac{K_t}{J_t} = \epsilon \frac{K_t}{J_t} \tag{B.1}
\]
\[
K_t = \left(\frac{Y_t}{A_t}\right) \varphi + \beta \theta E_t K_{t+1} \varphi_{t+1} \tag{B.2}
\]
\[
J_t = (1 - \tau_t) U_{c,t} Y_t + \beta \theta E_t J_{t+1} \varphi_{t+1} \tag{B.3}
\]
\[
(1 + \rho Q_t^M) \frac{b_t^{M-1}}{\pi_t} = Q_t^{M-1} b_t^M + \tau_t Y_t - Z_t - G_t \tag{B.4}
\]
\[
Q_t^M = \beta E_t \frac{U_{c,t+1}^M}{U_{c,t}} \frac{1}{\pi_t} (1 + \rho Q_t^M) \tag{B.5}
\]
\[
\Delta_t = (1 - \theta) \left[\frac{1 - \theta \pi_t^{e^{-1}}}{1 - \epsilon}\right]^{1+\varphi} + \theta \pi_t^{e(1+\varphi)} \Delta_{t-1} \tag{B.6}
\]

The first-order conditions are
\[
Y_t : U_{Y,t} - \lambda_1^2 (\varphi + 1) A_t^{(\varphi+1)} Y_t^\varphi - \lambda_2^2 (1 - \tau_t) (U_{c,t} + U_{c,c,t} Y_t) - \lambda_4^2 \tau_t
\]
\[
- \beta \sigma \lambda_5^2 \frac{1}{\pi_t} + \rho Q_t^{M-1} C_{t-1}^{(\varphi+1)} C_{t-1}^{\varphi} + \sigma \lambda_{-1}^2 \frac{1}{\pi_t} + \rho Q_t C_{t-1}^{(\varphi+1)} C_{t-1}^{\varphi} = 0
\]
\[
\pi_t : \lambda_1^2 \left[\frac{1 + \varphi}{1 - \epsilon} p(\pi_t)\right]^{1+\varphi} p'(\pi_t) - \lambda_2^2 \tau \epsilon (1 + \varphi) K_t \pi_t^{e(1+\varphi)-1} - \lambda_3^2 \theta (\epsilon - 1) J_t \pi_t^{e-2}
\]
\[
- \lambda_4^2 (1 + \rho Q_t^M) b_{t-1} \pi_t^{-2} + \lambda_5^2 \frac{U_{c,t}}{U_{c,t-1}} (1 + \rho Q_t^M) \pi_t^{-2}
\]
\[
- \lambda_6^2 [(1 - \theta) \frac{1 + \varphi}{1 - \epsilon} p(\pi_t)\right]^{1+\varphi} p'(\pi_t) + \theta \epsilon (1 + \varphi) \Delta_{t-1} \pi_t^{e(1+\varphi)-1}] = 0
\]
\[
\Delta_t : U_{\Delta,t} + \lambda_6^2 - E_t \lambda_6^2 \beta \theta \pi_t^{e(1+\varphi)} = 0
\]
\[
K_t : - \lambda_1^2 \left[\frac{1}{\epsilon} - \frac{1}{J_t}\right] + \lambda_2^2 - \lambda_3^2 \theta \pi_t^{e(1+\varphi)} = 0
\]
\[
J_t : \lambda_1^2 \left[\frac{1}{\epsilon} \frac{K_t}{J_t}\right] + \lambda_3^2 - \lambda_3^2 \theta \pi_t^{e-1} = 0
\]
\[
\tau_t : \lambda_2^2 U_{c,t} - \lambda_4^2 = 0
\]
\[
b_t : \lambda_2^2 - E_t \lambda_2^2 \frac{U_{c,t}}{U_{c,t-1}} = 0
\]
\[
Q_t^M : \lambda_1^2 \left(\frac{\rho b_{t-1}}{\pi_t} - b_t\right) + \lambda_5^2 - \lambda_5^2 \frac{\rho U_{c,t}}{U_{c,t-1}} = 0
\]

If we redefine \(\lambda_1^2 = \lambda_1^2, \lambda_2^2 = \lambda_2^2, \lambda_3^2 = \lambda_3^2, \lambda_4^2 = \lambda_4^2, \lambda_5^5 = \lambda_5^5, \lambda_6^6 = \lambda_6^6\), then the first-order conditions can be simplified as
The associated steady-state price of the long-term bond is given by

\[
\bar{Q}^M = \frac{\beta}{1 - \beta \rho}
\]

which is increasing in average maturity \(\rho\). The intuition is very straightforward, long-term debt yields more coupon payments and therefore demands higher price.

---

**B.1 Deterministic steady state** Using the optimal allocation that appendix B describes, in a steady state with zero net inflation, \(\bar{\pi} = 1\), we have

\[
\Delta = 1, \quad \frac{\bar{K}}{\bar{J}} = \frac{\epsilon - 1}{\epsilon}, \quad \bar{Q}^s = \beta \quad \frac{\bar{b}}{\bar{S}} = \frac{1 - \beta \rho}{1 - \beta}
\]

The associated steady-state price of the long-term bond is given by

\[
\bar{Q}^M = \frac{\beta}{1 - \beta \rho}
\]

where \(\bar{\lambda}_t^1\) through \(\bar{\lambda}_t^6\) are the Lagrange multipliers. Condition (B.13) implies that the evolution of the Lagrange multiplier corresponding to the government budget \(\bar{\lambda}_t^4\) obeys a martingale. Condition (B.14) connects \(\bar{\lambda}_t^4\) to the Lagrange multiplier corresponding to the maturity structure \(\bar{\lambda}_t^5\). Conditions (B.10) – (B.12) relate \(\bar{\lambda}_t^4\) to the Lagrange multiplier corresponding to the aggregate supply relations \(\bar{\lambda}_t^1\), \(\bar{\lambda}_t^2\), \(\bar{\lambda}_t^3\). Conditions (B.7) and (B.9) implicitly determine the shadow price of the each constraint. Notice that (B.9) determines the marginal utility loss from inflation, while (B.7) determines the marginal utility loss from output-gap.
The steady-state government budget constraint implies
\[ \bar{\tau} - s_g - s_z = (\beta^{-1} - 1) s_b \]
where \( s_b \equiv \bar{Q}M \bar{B}/\bar{Y} \) is the steady-state debt to GDP ratio, \( s_g \equiv \bar{G}/\bar{Y} \) is the steady state government purchases to GDP ratio, \( s_z \equiv \bar{Z}/\bar{Y} \) is the steady-state government transfers to GDP ratio.

Steady-state Lagrangian multipliers satisfy
\begin{align*}
\bar{\tilde{\lambda}}^1 &= (\theta - 1)\bar{\tilde{\lambda}}^3 \\
\bar{\tilde{\lambda}}^2 &= \frac{\epsilon}{1 - \epsilon} \bar{\tilde{\lambda}}^3 \\
\bar{\tilde{\lambda}}^3 &= \bar{\tilde{\lambda}}^4 \\
\bar{\tilde{\lambda}}^5 &= \bar{\tilde{\lambda}}^4 \\
(1 - \beta\theta)\bar{\tilde{\lambda}}^6 &= -U_{\Delta}(\bar{Y}, 1) \\
\left[ \frac{\epsilon(\phi + 1)}{1 - \epsilon} \right] \bar{Y}^{\phi} + \bar{U}_c + (1 - \bar{\tau})\bar{U}_{cc} \bar{Y} - \frac{(1 - \beta)\sigma}{1 - \beta\rho} \frac{s_b}{s_c} \bar{\tilde{\lambda}}^4 &= U_Y(\bar{Y}, 1)
\end{align*}

Note that \( \bar{\tilde{\lambda}}^4 \) and \( \bar{\tilde{\lambda}}^6 \) can be solved from (B.19) and (B.20), and at steady state the other multipliers are proportional to \( \bar{\tilde{\lambda}}^4 \). At steady state, the Lagrange multiplier associated with government budget therefore completely summarizes the distortions from output; the price dispersion summarizes the distortions from inflation.

**C Second-Order Approximation to Utility**

The life-time welfare of household is defined by
\[ U_0 = E_0 \sum_{t=0}^{\infty} \beta^t U(Y_t, \Delta_t, \xi_t) \quad (C.1) \]
where
\[ U(Y_t, \Delta_t, \xi_t) = \frac{(Y_t - G_t)^{1-\sigma}}{1 - \sigma} - \frac{(Y_t)^{1+\phi}}{1 + \phi} \Delta_t \\
= u(Y_t, G_t) - v(Y_t, \Delta_t, A_t) \quad (C.2) \]

We use a second-order Taylor expansion for a variable \( X_t \):
\[ X_t/\bar{X} = e^{ln X_t/\bar{X}} = e^{\hat{X}_t} = 1 + \hat{X}_t + \frac{1}{2} \hat{X}_t^2 \quad (C.3) \]
\[ \hat{X}_t = X_t - \bar{X} = \bar{X}(\hat{X}_t + \frac{1}{2} \hat{X}^2_t) \]  
(C.4)

The derivation of second-order approximation closely follows Benigno and Woodford (2004).

The first term in (C.2) can be approximated to second order as

\[
\begin{align*}
 u(Y_t, G_t) - \bar{u} &= \bar{u}_Y \hat{Y}_t + \bar{u}_G G_t + \bar{u}_{YG} \hat{Y} \hat{G}_t + \frac{1}{2} \bar{u}_{YY} \hat{Y}^2_t + \frac{1}{2} \bar{u}_{GG} \hat{G}^2_t + O(\|\xi_t\|^3) \\
 &= \hat{u}_Y \hat{Y}_t + \frac{1}{2} \bar{u}_Y \hat{Y}^2_t + \bar{u}_G G_t + \bar{u}_{YG} \hat{Y} \hat{G}_t + \frac{1}{2} \bar{u}_{YY} \hat{Y}^2_t + \frac{1}{2} \bar{u}_{GG} \hat{G}^2_t + O(\|\xi_t\|^3) \\
 &= \hat{u}_Y \hat{Y}_t + \frac{1}{2} \bar{u}_Y \hat{Y}^2_t + \bar{u}_{YG} \hat{Y} \hat{G}_t + \frac{1}{2} \bar{u}_{YY} \hat{Y}^2_t + O(\|\xi_t\|^3) + t.i.p. \\
 &= \hat{u}_Y \hat{Y}_t + \frac{1}{2} (1 - s_c^{-1}) \hat{Y}^2_t + s_g s_c^{-1} \sigma \hat{G}_t \hat{Y}_t + O(\|\xi_t\|^3) + t.i.p. \\
(C.5)
\end{align*}
\]

“t.i.p.” represents the terms that are independent of policy. The second term in (C.2) can be approximated by

\[
\begin{align*}
 v(Y_t, \Delta_t, A_t) - \bar{v} &= \bar{v}_Y \hat{Y}_t + \bar{v}_A \hat{A}_t + \bar{v}_Y \Delta \hat{A}_t + \bar{v}_A \Delta \hat{A}_t + \bar{v}_Y \Delta \hat{A}_t + \bar{v}_A \Delta \hat{A}_t \\
 &= \frac{1}{2} \bar{v}_Y \hat{Y}^2_t + \frac{1}{2} \bar{v}_A \hat{A}^2_t + \frac{1}{2} \bar{v}_Y \Delta \hat{A}_t^2 + O(\|\xi_t\|^3) \\
 &= \frac{1}{2} \bar{v}_Y \hat{Y}^2_t + \frac{1}{2} \bar{v}_A \hat{A}^2_t + \frac{1}{2} \bar{v}_Y \Delta \hat{A}_t^2 + O(\|\xi_t\|^3) + t.i.p. \\
 &= \frac{1}{2} \bar{v}_Y \hat{Y}^2_t + \frac{1}{2} \bar{v}_A \hat{A}^2_t + \frac{1}{2} \bar{v}_Y \Delta \hat{A}_t^2 + O(\|\xi_t\|^3) + t.i.p. \\
 &= \frac{1}{2} \bar{v}_Y \hat{Y}^2_t + \frac{1}{2} \bar{v}_A \hat{A}^2_t + \frac{1}{2} \bar{v}_Y \Delta \hat{A}_t^2 + O(\|\xi_t\|^3) + t.i.p. \\
 &= \frac{1}{2} \bar{v}_Y \hat{Y}^2_t + \frac{1}{2} \bar{v}_A \hat{A}^2_t + \frac{1}{2} \bar{v}_Y \Delta \hat{A}_t^2 + O(\|\xi_t\|^3) + t.i.p. \\
 &= \frac{1}{2} \bar{v}_Y \hat{Y}^2_t + \frac{1}{2} \bar{v}_A \hat{A}^2_t + \frac{1}{2} \bar{v}_Y \Delta \hat{A}_t^2 + O(\|\xi_t\|^3) + t.i.p. \\
(C.6)
\end{align*}
\]

From Benigno and Woodford (2004) we know that a second order approximation to (B.6) yields

\[
\hat{\Delta}_t = \theta \hat{\Delta}_{t-1} + \frac{\theta \epsilon}{1 - \theta}(1 + \epsilon \varphi)(1 + \epsilon \varphi) \frac{\bar{A}^2_t}{2} + O(\|\xi_t\|^3) + t.i.p. \\
(C.7)
\]

which implies that \( \hat{\Delta}_t = O(\pi^2_t) \).
Therefore (C.6) can be simplified as
\[
v(Y_t, \Delta_t, A_t) - \bar{v} = \bar{v}_Y [\dot{Y}_t + \frac{1}{2} (1 + \varphi) \ddot{Y}_t^2 + \frac{1}{1 + \varphi} \ddot{\Delta}_t - (1 + \varphi) \dot{Y}_t \dot{A}_t] + O(\|\xi_t\|^3) + t.i.p. \quad (C.8)
\]

Combine (C.5) and (C.8) and apply the relation \(\bar{v} = (1 - \Phi) \bar{u}_Y\), we approximate the life-time utility (C.1) as
\[
U_0 - \bar{U}_0 = \bar{u}_Y \bar{Y} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ A_y \dot{Y}_t + \frac{1}{2} A_{yy} \ddot{Y}_t^2 + A_{\xi} \dot{\xi}_t \dot{Y}_t - \frac{A_{\pi}}{2} \dot{\pi}_t^2 \right\} + O(\|\xi_t\|^3) + t.i.p. \quad (C.9)
\]

From Benigno and Woodford (2004) we observe
\[
E_0 \sum_{t=0}^{\infty} \beta^t \ddot{\Delta}_t = \frac{\theta \epsilon}{(1 - \theta)(1 - \beta \theta)} (1 + \varphi)(1 + \epsilon \varphi) \sum_{t=0}^{\infty} \beta^t \frac{\dot{\pi}_t^2}{2} \quad (C.10)
\]

Therefore, the second-order approximation to the life-time utility (C.1) can be further expressed as
\[
U_0 - \bar{U}_0 = \bar{u}_Y \bar{Y} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ A_y \dot{Y}_t + \frac{1}{2} A_{yy} \ddot{Y}_t^2 + A_{\xi} \dot{\xi}_t \dot{Y}_t - \frac{A_{\pi}}{2} \dot{\pi}_t^2 \right\} + O(\|\xi_t\|^3) + t.i.p. \quad (C.11)
\]

where
\[
A_y = \Phi
\]
\[
A_{yy} = (1 - \sigma s_c^{-1}) - (1 - \Phi)(1 + \varphi)
\]
\[
A_{\xi} = \sigma s_c^{-1} s_g \dot{G}_t + (1 - \Phi)(1 + \varphi) \dot{A}_t
\]
\[
A_{\pi} = (1 - \Phi) \frac{\theta \epsilon (1 + \epsilon \varphi)}{(1 - \theta)(1 - \beta \theta)}
\]

\(\Phi = 1 - (1 - \tau) \frac{\epsilon - 1}{\epsilon}\) measures the inefficiency of steady state of output. \(s_c = \frac{\bar{C}}{\bar{Y}}\) is steady state consumption to GDP ratio; \(s_g = \frac{\bar{G}}{\bar{Y}}\) is steady state government spending to GDP ratio.

**D Second-Order Approximation to Government’s IEC**

Recall the government budget constraint
\[
Q_t^M b_t^M + S_t = (1 + \rho Q_t^M) \frac{b_{t-1}^M}{\pi_t} \quad (D.1)
\]
and no-arbitrage condition

\[ Q^M_t = \beta E_t \frac{U_{c,t+1}}{\pi_{t+1}} \left( 1 + \rho Q^M_{t+1} \right) \]  \hspace{1cm} (D.2)

Define

\[ W_t = U_{c,t}(1 + \rho Q^M_t) b^{M-1}_t \frac{b^{M-1}_t}{\pi_t} \]  \hspace{1cm} (D.3)

By applying (D.2) and (D.3), (D.1) can be rewritten as

\[ W_t = U_{c,t} S_t + \beta E_t W_{t+1} \]  \hspace{1cm} (D.4)

and

\[ W_t = E_t \sum_{k=0}^{\infty} U_{c,t+k} S_{t+k} \]  \hspace{1cm} (D.5)

A second-order approximation to \( U_{c,t} S_t \) yields

\[ U_{c,t} S_t = \bar{U}_c S + \bar{U}_{cc} \tilde{C}_t + \bar{U}_{c} \tilde{S}_t + \frac{1}{2} \bar{S} \bar{U}_{ccc} \tilde{C}_t^2 + \bar{U}_{cc} \tilde{S}_t \tilde{C}_t \]  \hspace{1cm} (D.6)

We express \( \tilde{C}_t \) in terms of \( \tilde{Y}_t \) and \( \tilde{G}_t \) through the second order approximation to the identity \( C_t = Y_t - G_t \),

\[ \tilde{C}_t = \tilde{Y}(\tilde{Y}_t + \frac{1}{2} \tilde{Y}_t^2) - \tilde{G}(\tilde{G}_t + \frac{1}{2} \tilde{G}_t^2) \]  \hspace{1cm} (D.7)

Since \( S_t = \tau_t Y_t - G_t - Z_t \), a second-order approximation to the primary surplus can be written as

\[ \tilde{S}_t = \tilde{Y}(\tilde{\tau}_t + \tilde{Y}_t) + \frac{1}{2} \tilde{Y}(\tilde{\tau}_t^2 + \tilde{Y}_t^2) + \tilde{Y} \tilde{Y}_t \tilde{\tau}_t - G(\tilde{G}_t + \frac{1}{2} \tilde{G}_t^2) - Z(\tilde{Z}_t + \frac{1}{2} \tilde{Z}_t^2) \]  \hspace{1cm} (D.8)
Substituting (D.7) and (D.8) into (D.6), we obtain

\[
U_{c,t} = U_{c,t} = U_{c,\tau} S [Y_t \left( \frac{1}{2} \dot{Y}_t^2 \right) - \dot{G}(\dot{g}_t + \frac{1}{2} \dot{G}_t^2)]
+ \ddot{U}_c \tau \dot{Y}_t (\dot{t}_t - \dot{Y}_t) + \frac{1}{2} \ddot{Y}_t (\dot{t}_t^2 + \dot{Y}_t^2) + \ddot{Y}_t \dot{t}_t \dot{Y}_t - \ddot{G}(\dot{g}_t + \frac{1}{2} \dot{G}_t^2) - \ddot{Z}(\dot{Z}_t^2 + \frac{1}{2} \dot{Z}_t^2)
+ \frac{1}{2} S \dot{U}_{cc} (Y_t \dot{Y}_t^2 + \dot{G}_t \ddot{G}_t - 2 \dot{Y}_t \dot{G}_t)
+ \dot{U}_{cc} \left[ \ddot{Y}_t \dot{Y}_t^2 + \ddot{Y}_t \dot{Y}_t \dot{t}_t - (\dot{\tau} + 1) \dot{Y}_t \dot{G}_t \dot{G}_t - \ddot{Y}_t \dot{Y}_t \dot{G}_t + \ddot{G}_t G_t + \ddot{G}_t \ddot{G}_t \right]
+ \mathcal{O}(\|\xi_t\|^3)
\]

\[
= \ddot{U}_c S \left[ \left( -\frac{\sigma}{s_c} + \frac{\ddot{Y}}{s_c} \right) \dot{Y}_t + \frac{\ddot{Y}}{s_c} \ddot{\tau}_t + \frac{1}{2} \frac{\ddot{Y}}{s_c} \dot{\tau}_t^2 + \frac{1}{2} \frac{\ddot{Y}}{s_c} (1 - 2s_c) - \frac{\sigma}{s_c} \frac{2 \sigma}{s_c} + \frac{\sigma^2}{s_c} \ddot{\tau}_t + \frac{\ddot{Y}}{s_c} (1 - s_c^{-1}) \dot{\tau}_t \right]
+ \frac{\sigma s_g}{s_c} \ddot{Y} \dot{\tau}_t + \frac{\sigma s_g}{s_c} \ddot{Y} \ddot{\tau}_t - \frac{\sigma s_g}{s_c} \left( \frac{\sigma + 1}{s_c} \frac{\ddot{Y}}{s_c} - \ddot{Y} \dot{\tau}_t \ddot{\tau}_t - \ddot{Z} \dot{\tau}_t + (s_c^{-1} s_g - \frac{\ddot{G}}{s_c}) \dot{G}_t \right) + \mathcal{O}(\|\xi_t\|^3)
\]

Therefore, by substituting (D.9) into (D.5), we express the second-order approximation to government IEC as

\[
\frac{W_0 - \ddot{W}}{W} = (1 - \beta) E_0 \sum_{t=0}^{\infty} \beta^t \left[ B_y \dot{Y}_t + B_\tau \dot{\tau}_t + B_{yy} \dot{Y}_t^2 + \frac{1}{2} B_{\tau \tau} \dot{\tau}_t^2 + B_{\xi} \dot{\xi}_t \dot{Y}_t + B_{\xi \tau} \dot{\xi}_t \dot{\tau}_t + B_{\xi \xi} \dot{\xi}_t \dot{\xi}_t \right]
+ \mathcal{O}(\|\xi_t\|^3) + t.i.p.
\]

(D.10)

where

\[
B_y = -\frac{\sigma}{s_c} + \frac{\ddot{Y}}{s_c} \quad B_\tau = \frac{\ddot{Y}}{s_c} \quad B_{\xi} \dot{\xi}_t = -\frac{s_g}{s_d} \ddot{Z}_t + (\frac{s_c}{s_g} - \frac{s_d}{s_g}) \dot{G}_t
\]
\[
B_{yy} = \frac{\ddot{Y}}{s_c} (1 - 2s_c) - \frac{\sigma}{s_c} \frac{2 \sigma}{s_c} + \frac{\sigma^2}{s_c} \quad B_{\tau \tau} = \frac{\ddot{Y}}{s_c} (1 - 2s_c) - \frac{\sigma}{s_c} \frac{2 \sigma}{s_c} + \frac{\sigma^2}{s_c}
\]
\[
B_{\xi \tau} \dot{\xi}_t = \frac{s_g}{s_c} \ddot{Y} \dot{\tau}_t - \frac{s_g}{s_c} \left( \frac{\sigma + 1}{s_c} \frac{\ddot{Y}}{s_c} - \ddot{Y} \dot{\tau}_t \ddot{\tau}_t - \ddot{Z} \dot{\tau}_t + (s_c^{-1} s_g - \frac{\ddot{G}}{s_c}) \dot{G}_t \right)
\]
\[
B_{\xi \xi} \dot{\xi}_t = \frac{s_g}{s_c} \ddot{Y} \dot{\xi}_t
\]

\[
s_d = \frac{\ddot{Z}}{Y} \quad \text{is steady state government transfer payment to GDP ratio;} \quad s_d = \frac{\ddot{S}}{Y} \quad \text{is steady state surplus to GDP ratio.}
\]
E Second-Order Approximation to Aggregate Supply Relation

The aggregate supply relation is defined by the equations

\[
J_t \left[ \frac{1 - \theta \pi_t^{e-1}}{1 - \theta} \right] = \frac{\epsilon}{\epsilon - 1} K_t \tag{E.1}
\]

\[
K_t = (\frac{Y_t}{A_t})^{\varphi+1} + \beta \theta E_t K_{t+1} \pi_t^{e(1+\varphi)} \tag{E.2}
\]

\[
J_t = (1 - \tau_t)U_{c_t}Y_t + \beta \theta E_t J_{t+1} \pi_t^{e-1} \tag{E.3}
\]

A second-order approximation to (E.1) can be written as

\[
\frac{\epsilon}{\epsilon - 1} \tilde{K}_t - \tilde{J}_t = \tilde{J} \frac{\theta}{1-\theta} (1 + \epsilon \varphi) \{ \tilde{\pi}_t + \frac{1}{2} \left[ \frac{\theta}{1-\theta} \epsilon (\varphi + 1) + (\epsilon - 2) \right] \tilde{\pi}_t \} + \frac{\theta}{1-\theta} (1 + \epsilon \varphi) \tilde{J}_t \tilde{\pi}_t + \mathcal{O}(\| \xi_t \|^3) \tag{E.4}
\]

A second-order approximation to (E.2) can be written as

\[
\tilde{K}_t = \beta \theta E_t \tilde{K}_{t+1} + \beta \theta (1 + \varphi) \tilde{K} \{ E_t \tilde{\pi}_{t+1} + \frac{1}{2} \left[ \epsilon (1 + \varphi) - 1 \right] E_t \tilde{\pi}_{t+1} \} + \beta \theta (1 + \varphi) \tilde{K}_{t+1} \tilde{\pi}_{t+1} + (1 + \varphi) \tilde{Y}^\varphi \tilde{Y}_t - (1 + \varphi)^2 \tilde{Y}^\varphi \tilde{Y}_t^2 + \frac{1}{2} \epsilon (1 + \varphi) \tilde{Y}^\varphi-1 \tilde{Y}_t^2 - (\varphi + 1) \tilde{Y}^\varphi+1 \tilde{A}_t + \mathcal{O}(\| \xi_t \|^3) + \text{t.i.p.} \tag{E.5}
\]

A second-order approximation to (E.3) can be written as

\[
\tilde{J}_t = \beta \theta E_t \tilde{J}_{t+1} + \beta \theta (\epsilon - 1) \tilde{J} \{ E_t \tilde{\pi}_{t+1} + \frac{1}{2} \left[ \epsilon (1 + \varphi) - 1 \right] E_t \tilde{\pi}_{t+1} \} + \beta \theta (\epsilon - 1) \tilde{J}_{t+1} \tilde{\pi}_{t+1} + (1 - \tilde{\tau}_t)(\tilde{U}_c + \tilde{U}_{cc} \tilde{Y}) \tilde{Y}_t - \tilde{U}_c \tilde{Y} \tilde{\tau}_t + \frac{1}{2} \left[ (1 - \tilde{\tau}_t)(2 \tilde{U}_{cc} + \tilde{U}_{ccc} \tilde{Y}) \tilde{Y}_t^2 - (\tilde{U}_c + \tilde{U}_{cc} \tilde{Y}) \tilde{Y}_t \tilde{\tau}_t + \tilde{U}_{cc} \tilde{Y} \tilde{\tau}_t \tilde{G}_t - (1 - \tilde{\tau}_t)(\tilde{U}_c + \tilde{U}_{cc} \tilde{Y}) \tilde{Y}_t \tilde{G}_t + \mathcal{O}(\| \xi_t \|^3) + \text{t.i.p.} \tag{E.6}
\]

Therefore, \( \frac{\epsilon}{\epsilon - 1} \tilde{K}_t - \tilde{J}_t = \beta \theta E_t \left( \frac{\epsilon}{\epsilon - 1} \tilde{K}_{t+1} - \tilde{J}_{t+1} \right) \) can be expressed as

\[
\frac{\epsilon}{\epsilon - 1} \tilde{K}_t - \tilde{J}_t = \beta \theta E_t \left( \frac{\epsilon}{\epsilon - 1} \tilde{K}_{t+1} - \tilde{J}_{t+1} \right) + \frac{\epsilon}{\epsilon - 1} \beta \theta \epsilon (1 + \varphi) \tilde{K} \left[ E_t \tilde{\pi}_{t+1} + \frac{1}{2} \left[ \epsilon (1 + \varphi) - 1 \right] E_t \tilde{\pi}_{t+1} \right] - \beta \theta (\epsilon - 1) \tilde{J} \left[ E_t \tilde{\pi}_{t+1} + \frac{1}{2} \left[ \epsilon (1 + \varphi) - 1 \right] E_t \tilde{\pi}_{t+1} \right] + \frac{\epsilon}{\epsilon - 1} \beta \theta \epsilon (1 + \varphi) \tilde{K}_{t+1} \tilde{\pi}_{t+1} - \beta \theta (\epsilon - 1) \tilde{J}_{t+1} \tilde{\pi}_{t+1} + \frac{\epsilon}{\epsilon - 1} \left[ (1 + \varphi) \tilde{Y}^\varphi \tilde{Y}_t - (1 + \varphi)^2 \tilde{Y}^\varphi \tilde{Y}_t^2 + \frac{1}{2} \epsilon (1 + \varphi) \tilde{Y}^\varphi-1 \tilde{Y}_t^2 - (\varphi + 1) \tilde{Y}^\varphi+1 \tilde{A}_t \right] - \left[ (1 - \tilde{\tau}_t)(\tilde{U}_c + \tilde{U}_{cc} \tilde{Y}) \tilde{Y}_t - \tilde{U}_c \tilde{Y} \tilde{\tau}_t + \frac{1}{2} \left[ (1 - \tilde{\tau}_t)(2 \tilde{U}_{cc} + \tilde{U}_{ccc} \tilde{Y}) \tilde{Y}_t^2 - (\tilde{U}_c + \tilde{U}_{cc} \tilde{Y}) \tilde{Y}_t \tilde{\tau}_t \right] + \tilde{U}_{cc} \tilde{Y} \tilde{\tau}_t \tilde{G}_t - (1 - \tilde{\tau}_t)(\tilde{U}_c + \tilde{U}_{cc} \tilde{Y}) \tilde{Y}_t \tilde{G}_t + \mathcal{O}(\| \xi_t \|^3) + \text{t.i.p.} \tag{E.7}
\]
Then we plug (E.4) into (E.7) and obtain

\[
\ddot{J} = \frac{\theta}{1 - \theta}(1 + \epsilon \varphi)\{\ddot{\pi}_t + \frac{1}{2}\frac{\theta}{1 - \theta}(\epsilon(\varphi + 1) + (\epsilon - 2))\dot{\pi}_t^2\} + \frac{\theta}{1 - \theta}(1 + \epsilon \varphi)\dot{J}_t \ddot{\pi}_t
\]

\[
= \beta \theta \ddot{J} = \frac{\theta}{1 - \theta}(1 + \epsilon \varphi)\{E_t \ddot{\pi}_{t+1} + \frac{1}{2}\left[\frac{\theta}{1 - \theta}(\epsilon(\varphi + 1) + (\epsilon - 2))E_t \dot{\pi}_{t+1}^2\right] + \beta \theta \frac{\theta}{1 - \theta}(1 + \epsilon \varphi)\dot{J}_{t+1} \ddot{\pi}_{t+1}
\]

\[
+ \frac{\epsilon}{\epsilon - 1} \beta \theta (1 + \epsilon \varphi) \ddot{K} \{E_t \ddot{\pi}_{t+1} + \frac{1}{2}(\epsilon(1 + \varphi) - 1)E_t \dot{\pi}_{t+1}^2\} - \beta \theta (\epsilon - 1)\ddot{J} [E_t \ddot{\pi}_{t+1} + \frac{1}{2}(\epsilon - 2)E_t \dot{\pi}_{t+1}^2]
\]

\[
+ \beta \theta (1 + \varphi)\{\dot{J}_{t+1} \ddot{\pi}_{t+1} + \dot{J}_t \ddot{\pi}_{t+1}\} - \beta \theta (\epsilon - 1)\dot{J}_{t+1} \ddot{\pi}_{t+1}
\]

\[
+ \frac{\epsilon}{\epsilon - 1}(1 + \varphi)\dot{Y} \dot{Y}_t' - (1 + \varphi^2)\dot{Y} \dot{Y}_t' \dot{A}_t + \frac{1}{2}(1 + \varphi)\dot{Y} \dot{Y}_t' - (\varphi + 1)\dot{Y} \dot{Y}_t'
\]

\[
- [(1 - \tau)(\dot{U}_c + \ddot{U}_c \dot{Y})\dot{Y}_t - \ddot{U}_c \dot{Y}_t + \frac{1}{2}(1 - \tau)(2\ddot{U}_c + \ddot{U}_c \dot{Y})\dot{Y}_t^2 - (\ddot{U}_c + \ddot{U}_c \dot{Y})\dot{Y}_t \ddot{Y}_t]
\]

\[
- [\ddot{U}_c \ddot{Y}_t G_t - (1 - \tau)(\ddot{U}_c + \ddot{U}_c \dot{Y})\dot{Y}_t G_t - (1 - \tau)\ddot{Y} \ddot{U}_c G_t] + O(\|\xi_t\|^3) + t.i.p. \tag{E.8}
\]

Note that at steady state we have the relations \( \ddot{K} = \frac{\epsilon - 1}{\epsilon}, (1 - \beta \theta)\ddot{K} = \dot{Y} \varphi + 1 \) and \( (1 - \beta \theta)\ddot{J} = (1 - \tau)\ddot{U}_c \dot{Y} \), therefore (E.8) can be simplified as

\[
\frac{\theta}{1 - \theta}(1 + \epsilon \varphi)\ddot{\pi}_t + \frac{1}{2}\frac{\theta}{1 - \theta}(1 + \epsilon \varphi)\frac{1}{1 - \theta}(\epsilon(\varphi + 1))\ddot{\pi}_t^2 + \frac{\theta}{1 - \theta}(1 + \epsilon \varphi)(1 + \epsilon \varphi)\ddot{J}_t \ddot{\pi}_t
\]

\[
\beta \theta \frac{\theta}{1 - \theta}(1 + \epsilon \varphi)E_t \ddot{\pi}_{t+1} + \frac{1}{2}\frac{\theta}{1 - \theta}(1 + \epsilon \varphi)\beta \frac{1}{1 - \theta}(\epsilon(\varphi + 1))\ddot{\pi}_{t+1}^2 + \beta \theta \frac{\theta}{1 - \theta}(1 + \epsilon \varphi)\ddot{J}_{t+1} \ddot{\pi}_{t+1}
\]

\[
+ (1 - \beta \theta)\{(1 - \varphi)\dot{Y}_t - (1 + \varphi^2)\dot{Y}_t \dot{A}_t + \frac{1}{2}(1 + \varphi^2)\dot{Y}_t^2 - (\varphi + 1)\dot{A}_t]
\]

\[
- [(1 - \tau)(\ddot{U}_c + \ddot{U}_c \dot{Y})\dot{Y}_t - \ddot{U}_c \dot{Y}_t + \frac{1}{2}(1 - \tau)(2\ddot{U}_c + \ddot{U}_c \dot{Y})\dot{Y}_t^2 - (\ddot{U}_c + \ddot{U}_c \dot{Y})\dot{Y}_t \ddot{Y}_t]
\]

\[
- \frac{s_2}{s_c} \dot{w}_c \ddot{Y}_t G_t - [s(1 + \sigma) s_{c}^{-2} - \sigma s_{c}^{-1}] s_g \dot{Y}_t G_t + \frac{s_2}{s_c} \dot{G}_t + O(\|\xi_t\|^3) + t.i.p. \tag{E.9}
\]

Define \( V_t = \ddot{\pi}_t + \frac{1}{2}\frac{1}{1 - \theta}(\epsilon(\varphi + 1))\ddot{\pi}_t^2 + \ddot{J}_t \ddot{\pi}_t\), and substitue into (E.9), we obtain a recursive relation

\[
V_t = \kappa \{C_y \dot{Y}_t + C_{\tau} \dot{\tau}_t + C_{\gamma t} \dot{Y}_t \dot{\tau}_t + \frac{1}{2} C_{\gamma y} \dot{Y}_t^2 + \frac{1}{2} C_{\tau \tau} \ddot{\tau}_t^2 + C_{\gamma \tau} \dot{\gamma}_t \dot{\tau}_t + C_{\gamma} \dot{\gamma}_t \dot{\gamma}_t + \frac{C_{\gamma}}{2} \ddot{\gamma}_t^2 + C_{\gamma} \dot{\gamma}_t \dot{\gamma}_t\} + \beta E_t V_t + O(\|\xi_t\|^3) + t.i.p. \tag{E.10}
\]

where

\[
C_y = 1 \quad C_{\tau} = \psi \quad C_{\pi} = \frac{\epsilon(1 + \varphi)}{\kappa} \quad C_{\gamma y} = (2 + \varphi - \sigma s_{c}^{-1}) + \sigma(s_{c}^{-1} - s_{c}^{-2})(\varphi + \sigma s_{c}^{-1})^{-1}
\]

\[
C_{\gamma t} = (1 - \sigma s_{c}^{-1}) \psi \quad C_{\tau \tau} = \psi \quad C_{\gamma} = \frac{s_2}{s_c} - \sigma s_{c}^{-1} \dot{s}_g \dot{G}_t
\]

\[
C_{\gamma} \dot{\gamma}_t = \frac{\sigma^2 s_{c}^{-2} + \sigma s_{c}^{-2} - \sigma s_{c}^{-1}}{\sigma + \sigma s_{c}^{-1}} s_g \dot{G}_t - \frac{(1 + \varphi)^2}{\varphi + \sigma s_{c}^{-1}} \dot{A}_t
\]

\[
C_{\gamma} \dot{\gamma}_t = -\frac{\varphi + 1}{\varphi + \sigma s_{c}^{-1}} \dot{A}_t - \frac{s_2}{s_c} \frac{\sigma}{\varphi + \sigma s_{c}^{-1}} \dot{G}_t
\]
and

$$
\kappa = \frac{1 - \theta (1 - \beta \theta) (\varphi + \sigma s_c^{-1})}{1 + \epsilon \varphi}
$$

$$
w_T = \frac{\tau}{1 - \tau}
$$

$$
\psi = \frac{w_T}{\varphi + \sigma s_c^{-1}}
$$

Integrate (E.10) forward from \( t = 0 \), we have

$$
V_0 = E_0 \sum_{t=0}^{\infty} \beta^t \kappa \{ C_y \hat{Y}_t + C_{\tau} \hat{\tau}_t + C_{\tau \tau} \hat{\tau}_t \hat{\tau}_t + \frac{1}{2} C_{yy} \hat{Y}_t^2 + \frac{1}{2} C_{\tau \tau} \hat{\tau}_t^2 + C_{\xi_y} \hat{\xi}_t \hat{Y}_t + C_{\xi_{\tau}} \hat{\xi}_t \hat{\tau}_t + \frac{C_\pi}{2} \hat{\pi}_t^2 + C_{\xi'} \hat{\xi}_t \} 
\quad + O(||\xi_t||^3) + t.i.p.
$$

(E.11)

**F Quadratic Approximation to Objective Function**

Now we use a linear combination of (D.10) and (E.11) to eliminate the linear term in the second order approximation to the welfare measure. The coefficients \( \mu_B, \mu_C \) should satisfy

$$
\mu_B B_y + \mu_C C_y = -\Phi
$$

$$
\mu_B B_\tau + \mu_C C_\tau = 0
$$

The solution is

$$
\mu_B = \frac{\Phi w_T}{\Gamma}
$$

$$
\mu_C = -\frac{\Phi (1 + w_g) (\varphi + \sigma s_c^{-1})}{\Gamma}
$$

where \( w_g = \frac{G + Z}{S} \) is steady-state government outlays to surplus ratio, and satisfies \( 1 + w_g = \frac{\dot{Y}}{S} \).

\( \Gamma = \sigma s_c^{-1} w_T + (1 + w_g) (\varphi + \sigma s_c^{-1} - w_T) \).

Therefore, we can finally express the objective function in the linear quadratic form of

$$
(\bar{u}_Y \bar{Y})^{-1} (U_0 - \bar{U}_0) + \mu_B (1 - \beta)^{-1} \frac{W_0 - \bar{W}}{W} + \mu_C \kappa^{-1} V_0 = -E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{1}{2} q_Y (\hat{Y}_t - \hat{Y}_t^c)^2 + \frac{1}{2} q_{\pi} \hat{\pi}_t^2 \right] + O(||\xi_t||^3) + t.i.p.
$$

(F.1)
with
\[
q_Y = (1 - \Phi)(\varphi + \sigma s_c^{-1}) + \Phi(\varphi + \sigma s_c^{-1})(1 + w_g)(1 + \varphi) + \Phi \sigma s_c^{-1}(1 + w_g)(1 + w_r) - \Phi \sigma s_c^{-2}(1 + w_g + w_r) / \Gamma
\]
\[
q_\pi = \Phi(1 + w_g) \epsilon(1 + \varphi)(\varphi + \sigma s_c^{-1}) \kappa / \Gamma + (1 - \Phi) \epsilon(\varphi + \sigma s_c^{-1}) / \kappa
\]

and \( Y_t^e \) denotes the efficient level of output, which is exogenous and depends on the vector of exogenous shocks \( \xi_t \),
\[
Y_t^e = q_Y^{-1}(A' \hat{\xi}_t + \mu_B B' \hat{\xi}_t + \mu_C C' \hat{\xi}_t)
\]
\[
= q_A \hat{A}_t + q_G \hat{G}_t + q_Z \hat{Z}_t
\]

where
\[
q_A = q_Y^{-1}[(1 - \Phi)(1 + \varphi) + \Phi(1 + w_g)(1 + \varphi) \gamma + \Phi \sigma s_c^{-1}(1 + w_g)(1 + w_r)]
\]
\[
q_G = q_Y^{-1}[\sigma \frac{s_g}{s_c} - \sigma \frac{s_g}{s_c} \Phi w_r / \Gamma \left( \frac{\sigma + 1}{s_d} - \frac{1}{s_d} \right) + \sigma \frac{s_g}{s_c} \Phi(1 + w_g)(w_r + 1 - \frac{\sigma + 1}{s_c})]
\]
\[
q_Z = q_Y^{-1} \frac{\Phi w_r}{\Gamma} \left[ \sigma \frac{s_x}{s_c} \frac{1}{s_d} \right]
\]

G Coefficients of Analytical Solution in Section 5

Recall when \( \rho = 0 \), we express inflation and welfare relevant output-gap as
\[
\pi_t = -m_\pi (L_t^b - L_{t-1}^b)
\]
\[
x_t = m_x L_t^b + n_x L_{t-1}^b
\]

where
\[
m_\pi = \frac{1 - \beta}{\kappa \overline{\psi} s_d} + 1
\]
\[
m_x = \frac{1}{\lambda} \left[ \left( \frac{1}{\psi} - 1 \right) (1 - \beta) \overline{\psi} - \beta \frac{\sigma}{s_c} \right]
\]
\[
n_x = \frac{1}{\lambda s_c}
\]

The solution for \( L_t^b \) and \( b_t \) are given by
\[
L_t^b = \frac{m_b}{m_b + n_b} L_{t-1}^b - \frac{1}{m_b + n_b} [F_t + b_{t-1}]
\]
\[
b_t = - E_t F_{t+1} - n_b L_t^b
\]
where

\[ m_b = m_\pi + \frac{\sigma}{s_c} n_x + (1 - \beta) b_r \left( \frac{m_\pi}{\kappa \psi} - \frac{n_x}{\psi} \right) + (1 - \beta) b_x n_x \]  
\[ (G.1) \]

\[ n_b = (\psi^{-1} - 1) b_r (m_x + n_x) \]  
\[ (G.2) \]

When \( \rho = 1 \), we express inflation and welfare relevant output-gap as

\[ \pi_t = -\tilde{m}_\pi (L_t^b - L_{t-1}^b) - L_t^b \]

\[ x_t = \tilde{m}_x L_t^b \]

where

\[ \tilde{m}_\pi = \frac{1 - \beta}{\kappa \psi} \frac{\bar{\tau}}{s_d} \]

\[ \tilde{m}_x = \frac{1}{\lambda} \left( \frac{1}{\psi} - 1 \right) (1 - \beta) \frac{\bar{\tau}}{s_d} \]

The solution for \( L_t^b \) and \( b_t \) in this case are given by

\[ L_t^b = \frac{\tilde{m}_b}{\tilde{m}_b + \tilde{n}_b} L_{t-1}^b - \frac{1}{\tilde{m}_b + \tilde{n}_b} [F_t' + b_{t-1}] \]

\[ b_t = -E_t F_{t+1} - \tilde{n}_b L_t^b \]

where

\[ \tilde{m}_b = (1 + \tilde{m}_x)\tilde{m}_\pi \]  
\[ (G.3) \]

\[ \tilde{n}_b = (\psi^{-1} - 1) b_r \tilde{m}_x + \frac{1}{1 - \beta} + \tilde{m}_\pi \]  
\[ (G.4) \]

**H U.S. Data**

Unless otherwise noted, the following data are from the National Income and Product Accounts Tables released by the Bureau of Economic Analysis. All NIPA data are nominal and in levels.

**Consumption, \( C \).** Total personal consumption expenditures (Table 1.1.5, line 2).

**Government spending, \( G \).** Federal government consumption expenditures and gross investment (Table 1.1.5, line 22).

**GDP, \( Y \).** \( Y = C + G \).

**Total tax revenues, \( \tau Y \).** Federal current tax receipts (Table 3.2, line 2) plus contributions for government social insurance (Table 3.2, line 11) plus Federal income receipts on assets (Table 3.2, line 12).

**Total government transfers, \( Z \).** Federal current transfer payments (Table 3.2, line 22) minus Federal current transfer receipts (Table 3.2, line 16) plus Federal capital transfers payments (Table
3.2, line 43) minus Federal capital transfer receipts (Table 3.2, line 39) plus Federal subsidies (Table 3.2, line 32).


**Total factor productivity,** \( A \). Business sector total factor productivity, produced on 03-May-2013 by John Fernald/Kuni Natsuki, http://www.frbsf.org/economic-research/total-factor-productivity-tfp/. All published variables are log-differenced and annualized. To be consistent with model with fixed capital, we compute \( dA = dY - (dhours + dLQ) \), where \( dhours \) and \( dLQ \) are business sector hours and labor composition/quality actually used. Given \( dA_t = 400 * \log(A_t) - \log(A_{t-1}) \), we compute the annualized level of TFP, normalizing \( A_{1947Q1} = 1 \).

We use data from 1948Q1 to 2013Q1 to calibrate the model to U.S. data. For steady states, we use the sample means reported below. For the quarterly calibration, we multiply \( B/Y \) by
d

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
</tr>
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<tbody>
<tr>
<td>G/Y</td>
<td>0.129</td>
</tr>
<tr>
<td>( \tau )</td>
<td>0.240</td>
</tr>
<tr>
<td>B/Y</td>
<td>0.489</td>
</tr>
</tbody>
</table>

d4. Lump-sum transfers as a share of GDP adjust to satisfy the steady state government budget constraint.

To calibrate the exogenous processes, we apply a Hodrick and Prescott (1997) filter to time series on \( G_t, \tau_t, Z_t, \) and \( A_t \) and estimate AR(1) processes using the cyclical components of the filtered data, denoting those components by \( \hat{g}_t, \hat{\tau}_t, \hat{z}_t, \hat{a}_t \). Let the AR(1) be \( \hat{x}_t = \rho_x \hat{x}_{t-1} + \varepsilon_{xt} \) with standard error of estimate \( \sigma_{\varepsilon} \). Estimates appear below with standard errors in parentheses.

<table>
<thead>
<tr>
<th>Variable, ( x )</th>
<th>( \rho_x )</th>
<th>( \sigma_{\varepsilon} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{g}_t )</td>
<td>0.886</td>
<td>0.027</td>
</tr>
<tr>
<td></td>
<td>(0.029)</td>
<td></td>
</tr>
<tr>
<td>( \hat{\tau}_t )</td>
<td>0.782</td>
<td>0.029</td>
</tr>
<tr>
<td></td>
<td>(0.038)</td>
<td></td>
</tr>
<tr>
<td>( \hat{z}_t )</td>
<td>0.549</td>
<td>0.045</td>
</tr>
<tr>
<td></td>
<td>(0.051)</td>
<td></td>
</tr>
<tr>
<td>( \hat{a}_t )</td>
<td>0.786</td>
<td>0.008</td>
</tr>
<tr>
<td></td>
<td>(0.038)</td>
<td></td>
</tr>
</tbody>
</table>